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### SUMMARY

*Williams type modal solutions of the elementary and Timoshenko beam equations are presented for the transient response of several uniform beams to a general applied load. Example computations are shown for a free-free beam subject to various concentrated loads at its center. The discussion includes factors influencing the convergence of modal solutions and factors to be considered in a choice of beam theory. Results obtained by two numerical procedures, the traveling-wave method and Houbolt's method, are also presented and discussed.*

### INTRODUCTION

The problem of obtaining the response of elastic structures to rapidly applied loading is of continuing concern to the aircraft industry inasmuch as aircraft structures must withstand blasts, landing impacts, and a variety of other transient loads. In order to study the various factors involved in this problem, it is desirable to consider simplified structures for which thorough studies are possible. Among the simplest examples of continuous elastic structures are uniform beams. Consequently, beams have been the subject of a considerable number of transient response investigations, and a variety of solutions of particular beam problems are scattered throughout the existing literature. (See, for example, refs. 1 to 7. For an extensive bibliography, see ref. 7.)

It is the purpose of the present paper to provide a relatively complete source of useful modal solutions and to discuss the factors influencing the covergence of modal solutions and factors involved in the choice of the proper beam theory to be used in an analysis. To this end, a consistent presentation is made of Williams type modal solutions for the response to a completely general transient load of three pertinent uniform beams (a free-free beam with a concentrated mass as its center, a cantilever beam, and a simply supported beam). (Some duplication of the existing literature is included for completeness.) Solutions, based on both the elementary and Timoshenko beam theories, are obtained by a process which can be readily extended to the solution of problems with time-dependent boundary conditions. The application of the method is illustrated for the case of the free-free beam with a concentrated mass, and results for all the beams are summarized in tables I and II. In addition, some typical computed results are shown for a free-free beam subjected to various concentrated loadings.

Another purpose of the present paper is a critical discussion of two numerical procedures, the traveling-wave method (ref. 6) and Houbolt's method (ref. 8). The procedures are briefly described and computations made with both methods are compared with the modal results.

### SYMBOLS

| | |
|---|---|
| $A_s$ | effective shear-carrying area of cross section |
| $C$ | arbitrary constant |
| $c_1$ | propagation velocity of bending discontinuities, $\sqrt{\frac{EI}{mr^2}}$ |
| $c_2$ | propagation velocity of shear discontinuities, $\sqrt{\frac{A_sG}{m}}$ |
| $E$ | Young's modulus of elasticity |
| $f$ | applied concentrated load |
| $\bar{f}$ | dimensionless applied concentrated load, $\frac{fl^2}{EI}$ |

[1] Supersedes NACA Technical Note 4244 by Robert W. Leonard, 1958.

$G$ shear modulus of elasticity
$I$ moment of inertia of cross section
$i,j$ integers
$k$ dimensionless frequency parameter, $\omega l^2\sqrt{\frac{m}{EI}}$
$k_{RI}$ dimensionless rotary inertia parameter, $r/l$
$k_s$ dimensionless transverse shear parameter, $\frac{1}{l}\sqrt{\frac{EI}{A_sG}}$
$l$ length of beam (half-length in case of free-free beam)
$M$ bending moment (see fig. 1)
$\overline{M}$ dimensionless bending moment, $Ml/EI$
$\overline{M}_s$ dimensionless static bending moment
$m_i$ generalized mass
$m$ mass per unit length
$m_c$ concentrated mass
$\overline{m}_c$ ratio of the concentrated mass to total mass of the beam, $m_c/ml$
$P_i$ generalized force
$q$ applied distributed load (see fig. 1)
$\overline{q}$ dimensionless applied distributed load, $ql^3/EI$
$r$ cross-sectional radius of gyration
$t$ time
$V$ transverse shear force (see fig. 1)
$\overline{V}$ dimensionless transverse shear force, $Vl^2/EI$
$\overline{V}_s$ dimensionless static transverse shear force
$x$ coordinate along the beam
$y$ deflection (see fig. 1)
$\overline{y}$ dimensionless deflection, $y/l$
$\overline{y}_i$ dimensionless translational component of $i$th natural mode
$\overline{y}_s$ dimensionless static deflection
$\overline{y}_r$ dimensionless rigid-body translation

$$\alpha_i=k_i\sqrt{\frac{1}{2}\left[-(k_s^2+k_{RI}^2)+\sqrt{(k_s^2-k_{RI}^2)^2+\frac{4}{k_i^2}}\right]}$$

$$\beta_i=k_i\sqrt{\frac{1}{2}\left[(k_s^2+k_{RI}^2)+\sqrt{(k_s^2-k_{RI}^2)^2+\frac{4}{k_i^2}}\right]}$$

$$\gamma_i=\frac{\alpha_i^2+k_i^2k_s^2}{\beta_i^2-k_i^2k_s^2}$$

$\delta(\xi)$ Dirac delta function $\left(\delta(\xi)=0 \text{ for } \xi\neq 0;\ \int_{-\infty}^{\infty}\delta(\xi)d\xi=1\right)$
$\theta$ dummy variable of integration

$$\Lambda_i=\frac{\sin\beta_i-\frac{\beta_i}{\alpha_i}\sinh\alpha_i}{\cos\beta_i+\frac{1}{\gamma_i}\cosh\alpha_i}$$

$\xi$ dimensionless space coordinate, $x/l$
$\tau$ dimensionless time, $\frac{t}{l^2}\sqrt{\frac{EI}{m}}$
$\phi_i$ $i$th generalized coordinate
$\psi$ rotation of beam cross section
$\psi_i$ rotational component of $i$th natural mode
$\psi_s$ static rotation of beam cross section
$\omega$ circular frequency of natural vibration
$\mathbf{1}(\tau)$ step function ($\mathbf{1}(\tau)=0$ for $\tau<0$; $\mathbf{1}(\tau)=1$ for $\tau\geqq 0$)

Matrix notation:
[ ] rectangular matrix
⌊ ⌋ row matrix
{ } column matrix
⌈ ⌋ diagonal matrix

Primes and Roman numeral superscripts are used to denote partial differentiation with respect to $\xi$. Dots denote partial differentiation with respect to $\tau$.

## WILLIAMS TYPE MODAL SOLUTIONS

In normal-mode solutions for the response of beams to transient loads, the response is expanded in terms of a series of normal modes of the beam. The coefficients of the expansion (the generalized coordinates) are determined from the governing differential equations and the boundary and initial conditions. Williams type modal solutions (ref. 2) differ from ordinary normal-mode solutions by virtue of the isolation of that portion of the response which may be obtained in closed form by a process of direct integration—the so-called "static" portion of the response. Only the remaining "dynamic" portion of the response is expanded in series form.

The advantage of the Williams method over ordinary modal solutions is its ability to yield, for many loading conditions, a more accurate result with the same number of terms in the series. (See, for example, refs. 4 and 5.) It is particularly advantageous where the response function is discontinuous. (An example of this is the determination of the shear due to a concentrated load.) The discontinuity is contained exactly in the separated static portion of the response and the

series is only required to reproduce a continuous remainder.

In the Williams method, the isolated portion of the response is termed static because significant parts of the inertia forces are ignored in its determination. In general, however, it is time dependent by virtue of the time dependence of the applied load and of the nonhomogeneous time-dependent boundary conditions if such are imposed. In the case of beams with a fixed point of reference, such as cantilever or simply supported beams, all inertia forces are ignored in the determination of this static part of the response; for beams with rigid-body freedoms, however, the inertia forces due to the rigid-body motion must be taken into account.

One method of obtaining Williams type modal solutions is illustrated herein for both the elementary and Timoshenko beam theories.

### ELEMENTARY BEAM THEORY

**Basic equations.**—The motion of a beam subjected to an applied load of intensity $q(x,t)$ is usually taken to be governed by the Bernoulli-Euler equation

$$\frac{\partial^2}{\partial x^2} EI \frac{\partial^2 y}{\partial x^2} + m \frac{\partial^2 y}{\partial t^2} = q \tag{1}$$

where $x$ is the coordinate along the beam, $t$ is time, $y(x,t)$ is the deflection (see fig. 1), $EI(x)$ is the bending stiffness of the beam, and $m(x)$ is its mass per unit length. The internal bending moment $M(x,t)$ and the shear force $V(x,t)$ at any cross section (see fig. 1) are given by

$$M = -EI \frac{\partial^2 y}{\partial x^2} \tag{2}$$

and

$$V = -\frac{\partial}{\partial x} EI \frac{\partial^2 y}{\partial x^2} \tag{3}$$

respectively.

For a uniform beam, these equations may be written in the dimensionless forms

$$\bar{y}^{IV} + \ddot{\bar{y}} = \bar{q} \tag{4}$$

$$\overline{M} = -\bar{y}'' \tag{5}$$

$$\overline{V} = -\bar{y}''' \tag{6}$$

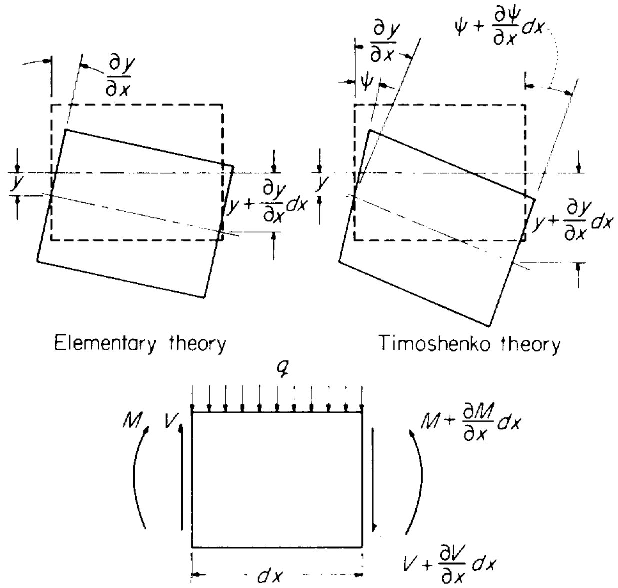



FIGURE 1.—Positive distortions and positive internal forces and moments associated with a typical beam element.

where

$$\bar{y}(\xi,\tau) = \frac{y}{l}$$

$$\overline{M}(\xi,\tau) = \frac{Ml}{EI}$$

$$\overline{V}(\xi,\tau) = \frac{Vl^2}{EI}$$

$$\bar{q}(\xi,\tau) = \frac{ql^3}{EI}$$

and $l$ is the length of the beam or half-length in the case of a free-free beam. The primes denote partial differentiation with respect to $\xi = \frac{x}{l}$ and dots denote partial differentiation with respect to $\tau = \frac{t}{l^2}\sqrt{\frac{EI}{m}}$.

**Symmetrical free-free beam with concentrated mass.**—For symmetrical motion of a uniform beam having free ends at $\xi = 1$ and $\xi = -1$, attention is restricted to the portion $0 \leqq \xi \leqq 1$ with boundary conditions stated in the form

$$\bar{y}'(0,\tau) = 0 \tag{7a}$$

$$\bar{y}'''(0,\tau) = 0 \tag{7b}$$

$$\bar{y}''(1,\tau)=0 \tag{7c}$$

$$\bar{y}'''(1,\tau)=0 \tag{7d}$$

If, in addition, the free-free beam has a concentrated mass $2m_c$ located at the center $\xi=0$, the influence of this mass may be introduced into the problem by changing the boundary condition, equation (7b), to $\bar{y}'''(0,\tau)+\bar{m}_c\ddot{\bar{y}}(0,\tau)=0$ where $\bar{m}_c=m_c/ml$. On the other hand, the boundary conditions, equations (7), may be left unchanged and the differential equation (eq. (4)) altered to

$$\bar{y}^{IV}+[1+\bar{m}_c\delta(\xi)]\ddot{\bar{y}}=\bar{q} \tag{8}$$

where $\delta(\xi)$ is the Dirac delta function. In the solution that follows, the latter alternative is chosen.

The beam is assumed to be initially at rest and undeflected; that is,

$$\bar{y}(\xi,0)=\dot{\bar{y}}(\xi,0)=0 \tag{9}$$

Then the response to a general symmetrical load $\bar{q}(\xi,\tau)$ may be obtained in the Williams form by the following procedure.

The solution is assumed in the form

$$\bar{y}(\xi,\tau)=\bar{y}_r(\tau)+\bar{y}_s(\xi,\tau)+\sum_{i=0}^{\infty}\phi_i(\tau)\bar{y}_i(\xi) \tag{10}$$

The quantity $\bar{y}_r$ is the rigid-body translation of the free-free beam. It is determined to satisfy the differential equation

$$\ddot{\bar{y}}_r(\tau)=\frac{1}{1+\bar{m}_c}\int_0^1\bar{q}(\xi,\tau)d\xi \tag{11}$$

and the initial conditions

$$\bar{y}_r(0)=\dot{\bar{y}}_r(0)=0 \tag{12}$$

The quantity $\bar{y}_s(\xi,\tau)$ is the static deflection determined to satisfy

$$\bar{y}_s^{IV}(\xi,\tau)=\bar{q}(\xi,\tau)-[1+\bar{m}_c\delta(\xi)]\ddot{\bar{y}}_r(\tau) \tag{13}$$

and the cantilever boundary conditions

$$\left.\begin{aligned}\bar{y}_s(0,\tau)&=0\\ \bar{y}_s'(0,\tau)&=0\\ \bar{y}_s''(1,\tau)&=0\\ \bar{y}_s'''(1,\tau)&=0\end{aligned}\right\} \tag{14}$$

Note that, by virtue of the definition of $\bar{y}_r$, $\bar{y}_s$ also satisfies the boundary condition $\bar{y}_s'''(0,\tau)=0$. Finally, the shapes $\bar{y}_i(\xi)$ (where $i=0, 1, 2, \ldots$) are the natural vibration modes of the beam satisfying

$$\bar{y}_i^{IV}(\xi)=[1+\bar{m}_c\delta(\xi)]k_i^2\bar{y}_i(\xi) \tag{15}$$

and the boundary conditions

$$\left.\begin{aligned}\bar{y}_i'(0)&=0\\ \bar{y}_i'''(0)&=0\\ \bar{y}_i''(1)&=0\\ \bar{y}_i'''(1)&=0\end{aligned}\right\} \tag{16}$$

where the dimensionless frequency coefficients $k_i$ (where $i=0, 1, 2, \ldots$) are defined by $k_i=\omega_i l^2\sqrt{\frac{m}{EI}}$. Further, it can be shown that the modes $\bar{y}_i(\xi)$ satisfy the orthogonality condition

$$\int_0^1[1+\bar{m}_c\delta(\xi)]\bar{y}_i(\xi)\bar{y}_j(\xi)d\xi=0 \quad (i\neq j) \tag{17}$$

Note that, by virtue of the arbitrary selection of a datum plane for $\bar{y}_s$, the dynamic portion of the response, in general, still contains a rigid-body component ($i=0$). As defined, the total deflection $\bar{y}(\xi,\tau)$ satisfies the boundary conditions (eqs. (7)). There remains the problem of determining the coefficients $\phi_i(\tau)$ so that the differential equation (eq. (8)) and initial conditions (eqs. (9)) are satisfied.

If expression (10) is substituted into differential equation (8) and equations (13) and (15) are taken into account, the differential equation is reduced to

$$\sum_{i=0}^{\infty}[\ddot{\phi}_i(\tau)+k_i^2\phi_i(\tau)]\bar{y}_i(\xi)=-\ddot{\bar{y}}_s(\xi,\tau) \tag{18}$$

Multiplying equation (18) by $[1+\bar{m}_c\,\delta(\xi)]\bar{y}_j(\xi)$ and integrating with respect to $\xi$ from 0 to 1 yields, in view of equation (17), the following result:

$$\ddot{\phi}_i(\tau)+k_i^2\phi_i(\tau)=-\frac{\ddot{P}_i(\tau)}{m_i} \qquad (i=0, 1, 2, \ldots) \tag{19}$$

where

$$m_i=\int_0^1[1+\bar{m}_c\delta(\xi)]\bar{y}_i^2(\xi)d\xi$$

$$P_i(\tau)=\int_0^1 [1+\overline{m}_c\delta(\xi)]\overline{y}_i(\xi)\overline{y}_s(\xi,\tau)d\xi$$

Similarly, substitution of expression (10) into the initial conditions (eq. (9)) and taking into account equations (12) lead to the following conditions:

$$\left.\begin{aligned}\phi_i(0)&=-\frac{P_i(0)}{m_i} \qquad (i=0,\ 1,\ 2,\ \ldots)\\ \dot{\phi}_i(0)&=-\frac{\dot{P}_i(0)}{m_i} \qquad (i=0,\ 1,\ 2,\ \ldots)\end{aligned}\right\} \tag{20}$$

A simple formula for the generalized-mass integral $m_i$ for $i=1,\ 2,\ \ldots$ has been presented in reference 9 for uniform beams having any of the usual end conditions (free, pinned, or clamped) but without concentrated masses. In terms of the dimensionless quantities defined herein, the extension of this formula to beams with a concentrated mass $\overline{m}_c$ at $\xi=0$ is

$$\begin{aligned}m_i&=\int_0^1 [1+\overline{m}_c\,\delta(\xi)]\,\overline{y}_i^2(\xi)d\xi\\ &=\frac{1}{4}\,\overline{m}_c\overline{y}_i^2(0)+\frac{1}{4k_i^2}\,[k_i^2\overline{y}_i^2(1)-2\overline{y}_i{}'(1)\overline{y}_i{}'''(1)\\ &\qquad +\overline{y}_i{}''^2(1)] \qquad (i=1,\ 2,\ \ldots)\end{aligned} \tag{21}$$

For the present case where the end ($\xi=1$) is free, equations (21) reduce to

$$m_i=\frac{1}{4}\,[\overline{m}_c\overline{y}_i^2(0)+\overline{y}_i^2(1)] \qquad (i=1,\ 2,\ \ldots) \tag{22a}$$

The rigid-body generalized mass ($i=0$) is seen to be

$$m_0=(1+\overline{m}_c)\overline{y}_0^2 \tag{22b}$$

Some reduction of the generalized-load integral $P_i(\tau)$ may also be accomplished in general terms for $i=1,\ 2,\ \ldots$. The quantity $[1+\overline{m}_c\delta(\xi)]\overline{y}_i(\xi)$ may be replaced by $\frac{1}{k_i^2}\overline{y}_i{}^{IV}(\xi)$ (eq. (15)); then, successive integrations by parts and application of the boundary conditions (eqs. (14) and (16)) reduce the integral to

$$P_i(\tau)=\frac{1}{k_i^2}\int_0^1 \overline{y}_s{}^{IV}(\xi,\tau)\overline{y}_i(\xi)d\xi \qquad (i=1,\ 2,\ \ldots)$$

Substituting from equation (13) and recalling that, in natural vibration, the inertia loads on a free-free beam are self-equilibrating yield, finally, the generalized load

$$P_i(\tau)=\frac{1}{k_i^2}\int_0^1 \overline{q}(\xi,\tau)\overline{y}_i(\xi)d\xi \qquad (i=1,\ 2,\ \ldots) \tag{23a}$$

On the other hand, for $i=0$, the quantity $P_0(\tau)$ is most simply expressed as

$$P_0(\tau)=\overline{y}_0\int_0^1 y_s(\xi,\tau)d\xi \tag{23b}$$

It might be noted here that, in the usual method of normal modes, the expressions for generalized force corresponding to equations (23a) do not have the factor $1/k_i^2$. This is one manifestation of the more rapid convergence of the Williams method.

The problem now requires direct integration of equations (11) and (13) for the deflections $\overline{y}_r$ and $\overline{y}_s$, solution of equations (19) for the generalized coordinates $\phi_i$, and solution of equation (15) for the natural modes of vibration $\overline{y}_i$, with each function satisfying the designated boundary or initial conditions. Direct integration of equation (11) with the initial conditions (eqs. (12)) taken into account yields

$$\overline{y}_r(\tau)=\frac{1}{1+\overline{m}_c}\int_0^\tau\int_0^\tau\int_0^1 \overline{q}(\xi,\tau)d\xi(d\tau)^2 \tag{24}$$

Substituting equation (11) into equation (13) and integrating four times, taking into account the boundary conditions on $\overline{y}_s$ (eqs. (14)), yields the following result:

$$\begin{aligned}\overline{y}_s(\xi,\tau)&=\int_0^\xi\int_0^\xi\int_1^\xi\int_1^\xi \overline{q}(\xi,\tau)\,(d\xi)^4\\ &\qquad -\frac{1}{1+\overline{m}_c}\left(\frac{\xi^4}{24}-\frac{\xi^3}{6}+\frac{\xi^2}{4}\right)\int_0^1\overline{q}(\xi,\tau)d\xi\end{aligned} \tag{25}$$

The solution of equation (19), satisfying also equations (20), is readily obtained by means of the Laplace transform. The result is

$$\phi_i(\tau)=-\frac{P_i(\tau)}{m_i}+\frac{k_i}{m_i}\int_0^\tau P_i(\theta)\sin k_i(\tau-\theta)d\theta$$
$$(i=0,\ 1,\ 2,\ \ldots) \tag{26}$$

Finally, the natural-mode shapes $\overline{y}_i$ and the corresponding frequency equation are derived in reference 10. These results, including the natural-

mode shapes and the frequency equation, are summarized for easy reference in table I(a).

Relations are also given in table I(a) for the moment $\overline{M}(\xi,\tau)$ and shear $\overline{V}(\xi,\tau)$ obtained by substitution of the deflection response into equations (5) and (6). (It is also possible to obtain these quantities by integrating the total load as

$$\overline{V}=\int_{\xi}^{1}(\overline{q}-\ddot{\overline{y}})d\xi$$

$$\overline{M}=-\int_{\xi}^{1}\overline{V}d\xi$$

However, some care must be exercised in using these formulas when the load function is discontinuous in time or has discontinuous first derivatives with respect to time.)

**Other configurations.**—The response of a uniform free-free beam without a concentrated mass is given by the results in table I(a) with $\overline{m}_c=0$. The response of a cantilever beam may also be obtained from the response of the free-free beam with the concentrated mass by a limiting process in which the mass $\overline{m}_c$ approaches infinity. Results for the cantilever beam are summarized in table I(b). For completeness, the Williams solution for a simply supported beam is shown in table I(c).

**Time-dependent boundary conditions.**—It is worthwhile to point out that the method outlined in this report is directly applicable to the solution of problems with nonhomogeneous time-dependent boundary conditions. Such problems require the separation of the solution into two parts, one satisfying the time-dependent boundary conditions and the other capable of being expanded in terms of time-independent functions such as the natural modes of the beam. (See, for example, ref. 11.) In the Williams method, this separation is already made and time-dependent boundary displacements or forces are simply introduced into the boundary conditions imposed on $\overline{y}_s$ or into the equations for rigid-body displacements.

Consider, for example, a uniform beam fixed at one end and given a variable displacement at the other, such that its differential equation and boundary conditions are

$$\overline{y}^{IV}(\xi,\tau)+\ddot{\overline{y}}(\xi,\tau)=\overline{q}(\xi,\tau)$$

$$\overline{y}(0,\tau)=\overline{y}'(0,\tau)=\overline{y}''(1,\tau)=0$$

$$\overline{y}(1,\tau)=g(\tau)$$

The solution would be assumed in the form of equation (10) but with $\overline{y}_r=0$ since there is no rigid-body translational freedom in this case. The static portion of the solution would be determined to satisfy

$$\overline{y}_s^{IV}(\xi,\tau)=\overline{q}(\xi,\tau)$$

and

$$\overline{y}_s(0,\tau)=\overline{y}_s'(0,\tau)=\overline{y}_s''(1,\tau)=0$$

$$\overline{y}_s(1,\tau)=g(\tau)$$

while the expansion functions $\overline{y}_i$ (where $i=1, 2, \ldots$) are the solutions of

$$\overline{y}_i^{IV}(\xi)=k_i^2\overline{y}_i(\xi)$$

and

$$\overline{y}_i(0)=\overline{y}_i'(0)=\overline{y}_i''(1)=\overline{y}_i(1)=0$$

(the natural modes of a clamped-pinned beam). In order to complete the solution, the generalized coordinates corresponding to a beam initially at rest and unstressed would have the usual form

$$\phi_i(\tau)=-\frac{P_i(\tau)}{m_i}+\frac{k_i}{m_i}\int_0^{\tau}P_i(\theta)\sin k_i(\tau-\theta)d\theta \qquad (i=1, 2, \ldots)$$

where

$$P_i(\tau)=\frac{1}{k_i^2}\int_0^1\overline{q}(\xi,\tau)\overline{y}_i(\xi)d\xi$$

$$m_i=\int_0^1\overline{y}_i^2(\xi)d\xi=-\frac{1}{2k_i^2}\overline{y}_i'(1)\overline{y}_i'''(1)$$

Similarly, a uniform free-free beam with a specified time-dependent displacement $g(\tau)$ at its center moves according to

$$\overline{y}^{IV}(\xi,\tau)+\ddot{\overline{y}}(\xi,\tau)=\overline{q}(\xi,\tau)$$

$$\overline{y}(0,\tau)=g(\tau)$$

$$\overline{y}'(0,\tau)=\overline{y}''(1,\tau)=\overline{y}'''(1,\tau)=0$$

In this case

$$\overline{y}_r(\tau)=g(\tau)$$

and $\overline{y}_s(\xi,\tau)$ is determined from

$$\overline{y}_s^{IV}(\xi,\tau)=\overline{q}(\xi,\tau)-\ddot{g}(\tau)$$

and

$$\overline{y}_s(0,\tau)=\overline{y}_s'(0,\tau)=\overline{y}_s''(1,\tau)=\overline{y}_s'''(1,\tau)=0$$

while the natural modes $\overline{y}_i$ (where $i=1, 2, \ldots$)

TABLE I.—RESPONSE OF A UNIFORM ELEMENTARY BEAM TO A GENERAL LOAD

(a) Symmetrical free-free beam with a concentrated mass

| Quantity | Analytical expression |
|---|---|
| $\bar{y}(\xi,\tau)$ | $\bar{y}_r(\tau)+\bar{y}_s(\xi,\tau)+\sum_{i=0}^{\infty}\phi_i(\tau)\,\bar{y}_i(\xi)$ |
| $\overline{M}(\xi,\tau)$ | $\overline{M}_s(\xi,\tau)+\sum_{i=1}^{\infty}\phi_i(\tau)\,\overline{M}_i(\xi)$ |
| $\overline{V}(\xi,\tau)$ | $\overline{V}_s(\xi,\tau)+\sum_{i=1}^{\infty}\phi_i(\tau)\,\overline{V}_i(\xi)$ |
| $\phi_i(\tau)$ | $-\frac{P_i(\tau)}{m_i}+\frac{k_i}{m_i}\int_0^{\tau}P_i(\theta)\sin k_i(\tau-\theta)\,d\theta$ |
| $P_i(\tau)$ | $\frac{1}{k_i^2}\int_0^1\bar{q}(\xi,\tau)\bar{y}_i(\xi)\,d\xi \qquad (i=1, 2, \ldots)$ <br> $\bar{y}_0\int_0^1\bar{y}_s(\xi,\tau)\,d\xi \qquad (i=0)$ |
| $m_i$ | $\frac{1}{4}[\overline{m}_c\,\bar{y}_i^2(0)+\bar{y}_i^2(1)] \qquad (i=1, 2, \ldots)$ <br> $(1+\overline{m}_c)\bar{y}_0^2 \qquad (i=0)$ |
| $\bar{y}_r(\tau)$ | $\frac{1}{1+\overline{m}_c}\int_0^{\tau}\int_0^{\tau}\int_0^1\bar{q}(\xi,\tau)d\xi(d\tau)^2$ |
| $\bar{y}_s(\xi,\tau)$ | $\int_0^{\xi}\int_0^{\xi}\int_1^{\xi}\int_1^{\xi}\bar{q}(\xi,\tau)(d\xi)^4-\frac{1}{1+\overline{m}_c}\left(\frac{\xi^4}{24}-\frac{\xi^3}{6}+\frac{\xi^2}{4}\right)\int_0^1\bar{q}(\xi,\tau)d\xi$ |
| $\overline{M}_s(\xi,\tau)$ | $-\int_1^{\xi}\int_1^{\xi}\bar{q}(\xi,\tau)(d\xi)^2+\frac{1}{1+\overline{m}_c}\left(\frac{\xi^2}{2}-\xi+\frac{1}{2}\right)\int_0^1\bar{q}(\xi,\tau)d\xi$ |
| $\overline{V}_s(\xi,\tau)$ | $-\int_1^{\xi}\bar{q}(\xi,\tau)d\xi+\frac{1}{1+\overline{m}_c}(\xi-1)\int_0^1\bar{q}(\xi,\tau)d\xi$ |
| $\bar{y}_i(\xi)$ | $C\Big[\cosh\sqrt{k_i}\cos\sqrt{k_i}\xi+\cos\sqrt{k_i}\cosh\sqrt{k_i}\xi$ <br> $-\frac{\overline{m}_c}{2}\sqrt{k_i}(\sin\sqrt{k_i}+\sinh\sqrt{k_i})(\cos\sqrt{k_i}+\cosh\sqrt{k_i})\left(\frac{\sin\sqrt{k_i}\xi-\sinh\sqrt{k_i}\xi}{\sin\sqrt{k_i}+\sinh\sqrt{k_i}}+\frac{\cosh\sqrt{k_i}\xi-\cos\sqrt{k_i}\xi}{\cos\sqrt{k_i}+\cosh\sqrt{k_i}}\right)\Big]$ |
| $\overline{M}_i(\xi)$ | $-Ck_i\Big[-\cosh\sqrt{k_i}\cos\sqrt{k_i}\xi+\cos\sqrt{k_i}\cosh\sqrt{k_i}\xi$ <br> $+\frac{\overline{m}_c}{2}\sqrt{k_i}(\sin\sqrt{k_i}+\sinh\sqrt{k_i})(\cos\sqrt{k_i}+\cosh\sqrt{k_i})\left(\frac{\sin\sqrt{k_i}\xi+\sinh\sqrt{k_i}\xi}{\sin\sqrt{k_i}+\sinh\sqrt{k_i}}-\frac{\cos\sqrt{k_i}\xi+\cosh\sqrt{k_i}\xi}{\cos\sqrt{k_i}+\cosh\sqrt{k_i}}\right)\Big]$ |
| $\overline{V}_i(\xi)$ | $-Ck_i^{3/2}\Big[\cosh\sqrt{k_i}\sin\sqrt{k_i}\xi+\cos\sqrt{k_i}\sinh\sqrt{k_i}\xi$ <br> $+\frac{\overline{m}_c}{2}\sqrt{k_i}(\sin\sqrt{k_i}+\sinh\sqrt{k_i})(\cos\sqrt{k_i}+\cosh\sqrt{k_i})\left(\frac{\cos\sqrt{k_i}\xi+\cosh\sqrt{k_i}\xi}{\sin\sqrt{k_i}+\sinh\sqrt{k_i}}+\frac{\sin\sqrt{k_i}\xi-\sinh\sqrt{k_i}\xi}{\cos\sqrt{k_i}+\cosh\sqrt{k_i}}\right)\Big]$ |

Frequency equation:

$$\cos\sqrt{k}\sinh\sqrt{k}+\sin\sqrt{k}\cosh\sqrt{k}+\overline{m}_c\sqrt{k}\,(1+\cos\sqrt{k}\cosh\sqrt{k})=0$$

TABLE I.—RESPONSE OF A UNIFORM ELEMENTARY BEAM TO A GENERAL LOAD—Continued

(b) Cantilever beam

| Quantity | Analytical expression |
| --- | --- |
| $\overline{y}(\xi,\tau)$ | $\overline{y}_s(\xi,\tau)+\sum_{i=1}^{\infty}\phi_i(\tau)\overline{y}_i(\xi)$ |
| $\overline{M}(\xi,\tau)$ | $\overline{M}_s(\xi,\tau)+\sum_{i=1}^{\infty}\phi_i(\tau)\overline{M}_i(\xi)$ |
| $\overline{V}(\xi,\tau)$ | $\overline{V}_s(\xi,\tau)+\sum_{i=1}^{\infty}\phi_i(\tau)\overline{V}_i(\xi)$ |
| $\phi_i(\tau)$ | $-\frac{P_i(\tau)}{m_i}+\frac{k_i}{m_i}\int_0^{\tau}P_i(\theta)\sin k_i(\tau-\theta)d\theta$ |
| $P_i(\tau)$ | $\frac{1}{k_i^2}\int_0^1\overline{q}(\xi,\tau)\overline{y}_i(\xi)d\xi$ |
| $m_i$ | $\int_0^1\overline{y}_i^2(\xi)d\xi=\frac{1}{4}\overline{y}_i^2(1)$ |
| $\overline{y}_s(\xi,\tau)$ | $\int_0^{\xi}\int_0^{\xi}\int_1^{\xi}\int_1^{\xi}\overline{q}(\xi,\tau)\,(d\xi)^4$ |
| $\overline{M}_s(\xi,\tau)$ | $-\int_1^{\xi}\int_1^{\xi}\overline{q}(\xi,\tau)\,(d\xi)^2$ |
| $\overline{V}_s(\xi,\tau)$ | $-\int_1^{\xi}\overline{q}(\xi,\tau)d\xi$ |
| $\overline{y}_i(\xi)$ | $C\left(\frac{\sin\sqrt{k_i}\xi-\sinh\sqrt{k_i}\xi}{\sin\sqrt{k_i}+\sinh\sqrt{k_i}}+\frac{\cosh\sqrt{k_i}\xi-\cos\sqrt{k_i}\xi}{\cos\sqrt{k_i}+\cosh\sqrt{k_i}}\right)$ |
| $\overline{M}_i(\xi)$ | $Ck_i\left(\frac{\sin\sqrt{k_i}\xi+\sinh\sqrt{k_i}\xi}{\sin\sqrt{k_i}+\sinh\sqrt{k_i}}-\frac{\cos\sqrt{k_i}\xi+\cosh\sqrt{k_i}\xi}{\cos\sqrt{k_i}+\cosh\sqrt{k_i}}\right)$ |
| $\overline{V}_i(\xi)$ | $Ck_i^{3/2}\left(\frac{\cos\sqrt{k_i}\xi+\cosh\sqrt{k_i}\xi}{\sin\sqrt{k_i}+\sinh\sqrt{k_i}}+\frac{\sin\sqrt{k_i}\xi-\sinh\sqrt{k_i}\xi}{\cos\sqrt{k_i}+\cosh\sqrt{k_i}}\right)$ |

Frequency equation:

$$1+\cos\sqrt{k}\cosh\sqrt{k}=0$$

TABLE I.—RESPONSE OF A UNIFORM ELEMENTARY BEAM TO A GENERAL LOAD—Concluded

(c) Simply supported beam

| Quantity | Analytical expression |
|---|---|
| $\bar{y}(\xi,\tau)$ | $\bar{y}_s(\xi,\tau)+\sum_{i=1}^{\infty}\phi_i(\tau)\sin i\pi\xi$ |
| $\overline{M}(\xi,\tau)$ | $\overline{M}_s(\xi,\tau)+\sum_{i=1}^{\infty}i^2\pi^2\phi_i(\tau)\sin i\pi\xi$ |
| $\overline{V}(\xi,\tau)$ | $\overline{V}_s(\xi,\tau)+\sum_{i=1}^{\infty}i^3\pi^3\phi_i(\tau)\cos i\pi\xi$ |
| $\phi_i(\tau)$ | $-2P_i(\tau)+2i^2\pi^2\int_0^{\tau}P_i(\theta)\sin i^2\pi^2(\tau-\theta)d\theta$ |
| $P_i(\tau)$ | $\frac{1}{i^4\pi^4}\int_0^1\bar{q}(\xi,\tau)\sin i\pi\xi\,d\xi$ |
| $\bar{y}_s(\xi,\tau)$ | $\int_0^{\xi}\int_0^{\xi}\int_0^{\xi}\int_0^{\xi}\bar{q}(\xi,\tau)\,(d\xi)^4-\xi\int_0^1\int_0^{\xi}\int_0^{\xi}\int_0^{\xi}\bar{q}(\xi,\tau)\,(d\xi)^4+\frac{\xi}{6}\,(1-\xi^2)\int_0^1\int_0^{\xi}\bar{q}(\xi,\tau)\,(d\xi)^2$ |
| $\overline{M}_s(\xi,\tau)$ | $-\int_0^{\xi}\int_0^{\xi}\bar{q}(\xi,\tau)\,(d\xi)^2+\xi\int_0^1\int_0^{\xi}\bar{q}(\xi,\tau)\,(d\xi)^2$ |
| $\overline{V}_s(\xi,\tau)$ | $-\int_0^{\xi}\bar{q}(\xi,\tau)\,d\xi+\int_0^1\int_0^{\xi}\bar{q}(\xi,\tau)\,(d\xi)^2$ |

are the modes of a cantilever beam. Or if, instead of $\bar{y}(0,\tau)=g(\tau)$, there is given the force boundary condition $\bar{y}'''(0,\tau)=h(\tau)$, the rigid-body motion is determined from

$$\ddot{\bar{y}}_r(\tau)=\int_0^1\bar{q}(\xi,\tau)d\xi-h(\tau)$$

The static solution $\bar{y}_s$ is taken to satisfy

$$\bar{y}_s^{IV}(\xi,\tau)=\bar{q}(\xi,\tau)-\ddot{\bar{y}}_r(\tau)$$

$$\bar{y}_s(0,\tau)=\bar{y}_s'(0,\tau)=\bar{y}_s''(1,\tau)=\bar{y}_s'''(1,\tau)=0$$

and the modes $\bar{y}_i$ (where $i=0, 1, 2, \ldots$) are the natural modes of a free-free beam. In this case, it can be shown by integrating the differential equation governing $\bar{y}_s$ that $\bar{y}_s'''(0,\tau)=h(\tau)$.

Thus, the treatment of problems with time-dependent boundary conditions involves no special separate procedure when the Williams' method is used.

### TIMOSHENKO'S BEAM THEORY

**Basic equations.**—In the elementary beam theory, deflection occurs only by virtue of the rotation of the beam elements and only their translational inertia is taken into account. The Timoshenko beam theory (ref. 9) permits additional deflection due to transverse shear and

accounts also for the rotational inertia of the beam elements. According to this theory, the motion of a beam subjected to an applied load of intensity $q(x,t)$ is governed by the equations (see, for example, ref. 6):

$$\left.\begin{aligned} &\frac{\partial}{\partial x} EI \frac{\partial \psi}{\partial x}+A_sG\left(\frac{\partial y}{\partial x}-\psi\right)-mr^2 \frac{\partial^2 \psi}{\partial t^2}=0 \\ &\frac{\partial}{\partial x}\left[A_sG\left(\frac{\partial y}{\partial x}-\psi\right)\right]-m \frac{\partial^2 y}{\partial t^2}=-q \end{aligned}\right\} \qquad (27)$$

where $\psi$ is the rotation of the cross section (see fig. 1), $r$ is the radius of gyration of the cross section, and $A_sG$ is the stiffness in transverse shear. The effective shear-carrying area $A_s$ differs from the total area because the shear stress is not constant over the cross section. The bending moment $M$ and transverse shear force $V$ are given by

$$M=-EI \frac{\partial \psi}{\partial x} \qquad (28)$$

and

$$V=A_sG\left(\frac{\partial y}{\partial x}-\psi\right) \qquad (29)$$

For uniform beams, these equations may be written in the dimensionless forms

$$\left.\begin{aligned} &\psi''+\frac{1}{k_s^2}(\overline{y}'-\psi)-k_{RI}^2\ddot{\psi}=0 \\ &\frac{1}{k_s^2}(\overline{y}'-\psi)'-\ddot{\overline{y}}=-\overline{q} \end{aligned}\right\} \qquad (30)$$

$$\overline{M}=-\psi' \qquad (31)$$

$$\overline{V}=\frac{1}{k_s^2}(\overline{y}'-\psi) \qquad (32)$$

where the transverse shear coefficient $k_s=\frac{1}{l}\sqrt{\frac{EI}{A_sG}}$ is a measure of the freedom of the beam to deflect through transverse shearing action and the rotary inertia coefficient $k_{RI}=\frac{r}{l}$ is a measure of the rotational inertia per unit length.

Note that the functions $\overline{y}$ and $\psi$ are both necessary for adequate definition of the deformation of the beam. Since these go hand-in-hand, the terms "solution" and "response," as used herein, will apply to these functions collectively and the single notation $\overline{y}(\xi,\tau);\psi(\xi,\tau)$ will be used to specify both functions.

**Symmetrical free-free beam with concentrated mass.**—For the application of Timoshenko's theory to the symmetrical motion of a uniform free-free beam with a mass $2\overline{m}_c$ at the center $\xi=0$, attention will again be restricted to the portion $0\leqq\xi\leqq1$. As in the case of the elementary theory, the effect of the mass may be introduced into the differential equations if desired. However, for illustrative purposes, the differential equations (30) will be left unchanged and the mass will be introduced in the boundary conditions; the boundary conditions then become

$$\left.\begin{aligned} &\psi(0,\tau)=0 \\ &\psi'(1,\tau)=0 \\ &\overline{y}'(1,\tau)-\psi(1,\tau)=0 \\ &\frac{1}{k_s^2}\overline{y}'(0,\tau)=\overline{m}_c\ddot{\overline{y}}(0,\tau) \end{aligned}\right\} \qquad (33)$$

Note that the location of the concentrated mass at $\xi=0$ and the restriction to symmetrical motion exclude any effect of the rotational inertia of the concentrated mass.

The beam is assumed initially at rest and undeflected; hence,

$$\left.\begin{aligned} &\overline{y}(\xi,0)=0 \\ &\dot{\overline{y}}(\xi,0)=0 \\ &\psi(\xi,0)=0 \\ &\dot{\psi}(\xi,0)=0 \end{aligned}\right\} \qquad (34)$$

With the problem thus completely defined by the differential equations (eqs. (30)), the boundary conditions (eqs. (33)), and the initial conditions (eqs. (34)), the solution may be obtained as follows

Assume that

$$\left.\begin{aligned} &\overline{y}(\xi,\tau)=\overline{y}_r(\tau)+\overline{y}_s(\xi,\tau)+\sum_{i=0}^{\infty}\phi_i(\tau)\overline{y}_i(\xi) \\ &\psi(\xi,\tau)=\psi_s(\xi,\tau)+\sum_{i=0}^{\infty}\phi_i(\tau)\psi_i(\xi) \end{aligned}\right\} \qquad (35)$$

where $\overline{y}_r(\tau)$ is again the rigid-body translation of the beam, $\overline{y}_s(\xi,\tau);\psi_s(\xi,\tau)$ is the static solution, and $\overline{y}_i(\xi);\psi_i(\xi)$ (where $i=0, 1, 2, \ldots$) are the natural vibration modes. The rigid-body translation of the beam $\overline{y}_r$ is governed again by the differential equation

$$\ddot{\overline{y}}_r=\frac{1}{1+\overline{m}_c}\int_0^1 \overline{q}(\xi,\tau)d\xi \qquad (36)$$

and the initial conditions

$$\overline{y}_r(0)=\dot{\overline{y}}_r(0)=0 \tag{37}$$

The static solution is determined to satisfy

$$\left.\begin{aligned}\psi_s''+\frac{1}{k_s^2}(\overline{y}_s'-\psi_s)=0\\ \frac{1}{k_s^2}(\overline{y}_s'-\psi_s)'=-\overline{q}+\ddot{\overline{y}}_r\end{aligned}\right\} \tag{38}$$

and the cantilever boundary conditions

$$\left.\begin{aligned}&\overline{y}_s(0,\tau)=0\\ &\psi_s(0,\tau)=0\\ &\psi_s'(1,\tau)=0\\ &\overline{y}_s'(1,\tau)-\psi_s(1,\tau)=0\end{aligned}\right\} \tag{39}$$

The mode shapes $\overline{y}_i(\xi)$; $\psi_i(\xi)$ satisfy

$$\left.\begin{aligned}\psi_i''+\frac{1}{k_s^2}(\overline{y}_i'-\psi_i)+k_i^2k_{RI}^2\psi_i=0\\ \frac{1}{k_s^2}(\overline{y}_i'-\psi_i)'+k_i^2\overline{y}_i=0\end{aligned}\right\} \tag{40}$$

the boundary conditions

$$\left.\begin{aligned}&\psi_i(0)=0\\ &\psi_i'(1)=0\\ &\overline{y}_i'(1)-\psi_i(1)=0\\ &\frac{1}{k_s^2}\overline{y}_i'(0)=-\overline{m}_ck_i^2\overline{y}_i(0)\end{aligned}\right\} \tag{41}$$

and the orthogonality relation

$$\int_0^1\{[1+\overline{m}_c\delta(\xi)]\overline{y}_i(\xi)\overline{y}_j(\xi)+k_{RI}^2\psi_i(\xi)\psi_j(\xi)\}d\xi=0 \qquad (i\neq j) \tag{42}$$

The derivation of this orthogonality relation is shown in the appendix along with the solutions to equations (40) and (41).

Substituting equations (35) into the differential equations (eqs. (30)) and utilizing equations (38) and (40) reduce the differential equations to

$$\sum_{i=0}^{\infty}(\ddot{\phi}_i+k_i^2\phi_i)\overline{y}_i=-\ddot{\overline{y}}_s \tag{43a}$$

$$\sum_{i=0}^{\infty}(\ddot{\phi}_i+k_i^2\phi_i)\psi_i=-\ddot{\ }_s \tag{43b}$$

Multiplying equation (43a) by $[1+\overline{m}_c\delta(\xi)]\overline{y}_j$ and equation (43b) by $k_{RI}^2\psi_j$, adding the two equations and integrating the sum over the range $0\leqq\xi\leqq 1$ yield the result

$$\ddot{\phi}_i+k_i^2\phi_i=-\frac{\ddot{P}_i(\tau)}{m_i} \qquad (i=0, 1, 2, \ldots) \tag{44}$$

which takes into account the orthogonality relation (eq. (42)). The generalized mass and generalized load appearing in equation (44) are

$$m_i=\int_0^1\left\{[1+\overline{m}_c\delta(\xi)]\overline{y}_i^2+k_{RI}^2\psi_i^2\right\}d\xi$$

and

$$P_i(\tau)=\int_0^1\left\{[1+\overline{m}_c\delta(\xi)]\overline{y}_i\overline{y}_s+k_{RI}^2\psi_i\psi_s\right\}d\xi$$

respectively.

By a similar process the initial conditions (eqs. (34)) become

$$\left.\begin{aligned}\phi_i(0)=-\frac{P_i(0)}{m_i} \qquad (i=0, 1, 2, \ldots)\\ \dot{\phi}_i(0)=-\frac{\dot{P}_i(0)}{m_i} \qquad (i=0, 1, 2, \ldots)\end{aligned}\right\} \tag{45}$$

It is shown in the appendix that the generalized masses $m_i$ of the given free-free beam can be evaluated for $i=1, 2, \ldots$ as follows:

$$m_i=-\frac{\psi_i(1)}{2k_i}\left[\frac{\partial}{\partial k}\psi'(1)\right]_{k=k_i} \qquad (i=1, 2, \ldots) \tag{46a}$$

The remaining generalized mass $m_0$ reduces, as in the elementary theory, to

$$m_0=(1+\overline{m}_c)\overline{y}_0^2 \tag{46b}$$

since the symmetric rigid-body mode has no rotational component $\psi_0$. Further, the generalized force $P_i(\tau)$ for $i=1, 2, \ldots$ may be reduced, by a process of substitution from equations (38) and (40) and integration by parts, to

$$P_i(\tau)=\frac{1}{k_i^2}\int_0^1\overline{y}_i(\xi)\overline{q}(\xi,\tau)d\xi \qquad (i=1, 2, \ldots) \tag{47a}$$

and the rigid-body generalized force $P_0$ is

$$P_0(\tau)=\overline{y}_0\int_0^1\overline{y}_s(\xi,\tau)d\xi \tag{47b}$$

On the basis of the assumed form of the solution expressed in equations (35), the problem of determining $y(\xi,\tau)$; $\psi(\xi,\tau)$ has been replaced by a number

of component problems requiring determination of the functions $\bar{y}_r(\tau)$, $\bar{y}_s(\xi,\tau);\psi_s(\xi,\tau)$, $\phi_i(\tau)$, and $\bar{y}_i(\xi);\psi_i(\xi)$. The solutions of these component problems must now be obtained.

For $\bar{y}_r(\tau)$, integration of equation (36) in conjunction with equations (37) yields

$$\bar{y}_r(\tau)=\frac{1}{1+\bar{m}_c}\int_0^\tau\int_0^\tau\int_0^1 \bar{q}(\xi,\tau)d\xi\,(d\tau)^2 \qquad (48)$$

For $\bar{y}_s(\xi,\tau);\psi_s(\xi,\tau)$, substituting $\ddot{\bar{y}}_r$ from equation (36) into equations (38) and integrating, in conjunction with the boundary conditions (eqs. (39)), gives

$$\left.\begin{aligned}\bar{y}_s(\xi,\tau)=&\int_0^\xi\int_0^\xi\int_1^\xi\int_1^\xi \bar{q}(\xi,\tau)\,(d\xi)^4-k_s^2\int_0^\xi\int_1^\xi \bar{q}(\xi,\tau)\,(d\xi)^2\\&+\frac{1}{1+\bar{m}_c}\left[k_s^2\left(\frac{\xi^2}{2}-\xi\right)-\left(\frac{\xi^4}{24}-\frac{\xi^3}{6}+\frac{\xi^2}{4}\right)\right]\int_0^1 \bar{q}(\xi,\tau)d\xi\\ \psi_s(\xi,\tau)=&\int_0^\xi\int_1^\xi\int_1^\xi \bar{q}(\xi,\tau)\,(d\xi)^3-\frac{1}{1+\bar{m}_c}\left(\frac{\xi^3}{6}-\frac{\xi^2}{2}+\frac{\xi}{2}\right)\int_0^1 \bar{q}(\xi,\tau)d\xi\end{aligned}\right\} \qquad (49)$$

Since equations (44) and (45) are identical to equations (19) and (20) of the elementary solution, the generalized coordinates $\phi_i(\tau)$ are again given by equation (26). The solution of equations (40) for the natural modes $\bar{y}_i(\xi);\psi_i(\xi)$ is given in the appendix.

The solution just obtained and corresponding solutions for cantilever and simply supported beams are summarized in table II.

## TWO NUMERICAL METHODS OF SOLUTION

### TRAVELING-WAVE METHOD

A traveling-wave method for calculating the response of a structure to transient loads can be devised only if the motion of the structure is governed by differential equations of the hyperbolic type. The simplest beam theory which completely fulfills this requirement is Timoshenko's theory, which includes the effects of both transverse shear and rotary inertia.

In developing a traveling-wave method of solution, it is convenient to first replace the Timoshenko partial differential equations (eqs. (27)) with the following four equivalent ordinary differential equations written along four characteristic lines I+, I−, II+, and II− in the $x,t$ plane:

Along I+ where $\frac{dt}{dx}=\frac{1}{c_1}$: $\quad \frac{1}{c_1}dM+mr^2d\Omega-Vdt=0 \qquad$ (50a)

Along I− where $\frac{dt}{dx}=-\frac{1}{c_1}$: $\quad \frac{1}{c_1}dM-mr^2d\Omega+Vdt=0 \qquad$ (50b)

Along II+ where $\frac{dt}{dx}=\frac{1}{c_2}$: $\quad \frac{1}{c_2}dV-mdv+(mc_2\Omega+q)dt=0 \qquad$ (50c)

Along II− where $\frac{dt}{dx}=-\frac{1}{c_2}$: $\quad \frac{1}{c_2}dV+mdv+(mc_2\Omega-q)dt=0 \qquad$ (50d)

The derivation of equations (50) is given in reference 6. The dependent variables are the moment $M$, the shear $V$, and the linear and angular velocities $v=\frac{\partial y}{\partial t}$ and $\Omega=\frac{\partial \psi}{\partial t}$. The quantities $c_1=\sqrt{\frac{EI}{mr^2}}$ and $c_2=\sqrt{\frac{A_sG}{m}}$ are the propagation velocities of discontinuities in moment and shear, respectively (phase velocities of disturbances with infinitesimal wave length). In each equation, the total differentials specify infinitesimal differences in the designated characteristic direction.

For any given beam, the slopes of the characteristic lines are known; hence, closely spaced networks of characteristic lines may be drawn in the space-time plane. Various schemes for the approximate step-by-step integration of equations (50) over such networks are possible. In general, all require some form of interpolation since Timoshenko's equations have two characteristic nets. (The particular case where the two nets coincide, $c_1=c_2$, has been treated in detail for uniform beams in ref. 6.) One integration scheme is briefly described in this section. Attention is restricted to a uniform beam for which the characteristics are straight lines.

TABLE II.—RESPONSE OF A UNIFORM TIMOSHENKO BEAM TO A GENERAL LOAD

(a) Symmetrical free-free beam with a concentrated mass

| Quantity | Analytical expression |
|---|---|
| $\overline{y}(\xi,\tau)$ | $\overline{y}_r(\tau)+\overline{y}_s(\xi,\tau)+\sum_{i=0}^{\infty}\phi_i(\tau)\overline{y}_i(\xi)$ |
| $\psi(\xi,\tau)$ | $\psi_s(\xi,\tau)+\sum_{i=1}^{\infty}\phi_i(\tau)\psi_i(\xi)$ |
| $\overline{M}(\xi,\tau)$ | $\overline{M}_s(\xi,\tau)+\sum_{i=1}^{\infty}\phi_i(\tau)\overline{M}_i(\xi)$ |
| $\overline{V}(\xi,\tau)$ | $\overline{V}_s(\xi,\tau)+\sum_{i=1}^{\infty}\phi_i(\tau)\overline{V}_i(\xi)$ |
| $\phi_i(\tau)$ | $-\frac{P_i(\tau)}{m_i}+\frac{k_i}{m_i}\int_0^{\tau}P_i(\theta)\sin k_i(\tau-\theta)d\theta$ |
| $P_i(\tau)$ | $\frac{1}{k_i^2}\int_0^1\overline{q}(\xi,\tau)\overline{y}_i(\xi)d\xi \qquad (i=1,2,\ .\ .\ .\ )$<br>$\overline{y}_0\int_0^1\overline{y}_s(\xi,\tau)d\xi \qquad (i=0)$ |
| $m_i$ | $-\frac{1}{2k_i}\psi_i(1)\left[\frac{\partial}{\partial k}\psi'(1)\right]_{k=k_i} \qquad (i=1,2,\ .\ .\ .\ )$<br>$(1+\overline{m}_c)\overline{y}_0^2 \qquad (i=0)$ |
| $\overline{y}_r(\tau)$ | $\frac{1}{1+\overline{m}_c}\int_0^{\tau}\int_0^{\tau}\int_0^1\overline{q}(\xi,\tau)d\xi(d\tau)^2$ |
| $\overline{y}_s(\xi,\tau)$ | $\int_0^{\xi}\int_0^{\xi}\int_1^{\xi}\int_1^{\xi}\overline{q}(\xi,\tau)(d\xi)^4-k_s^2\int_0^{\xi}\int_1^{\xi}\overline{q}(\xi,\tau)(d\xi)^2+\frac{1}{1+\overline{m}_c}\left[k_s^2\left(\frac{\xi^2}{2}-\xi\right)-\left(\frac{\xi^4}{24}-\frac{\xi^3}{6}+\frac{\xi^2}{4}\right)\right]\int_0^1\overline{q}(\xi,\tau)d\xi$ |
| $\psi_s(\xi,\tau)$ | $\int_0^{\xi}\int_1^{\xi}\int_1^{\xi}\overline{q}(\xi,\tau)(d\xi)^3-\frac{1}{1+\overline{m}_c}\left(\frac{\xi^3}{6}-\frac{\xi^2}{2}+\frac{\xi}{2}\right)\int_0^1\overline{q}(\xi,\tau)d\xi$ |
| $\overline{M}_s(\xi,\tau)$ | $-\int_1^{\xi}\int_1^{\xi}\overline{q}(\xi,\tau)(d\xi)^2+\frac{1}{1+\overline{m}_c}\left(\frac{\xi^2}{2}-\xi+\frac{1}{2}\right)\int_0^1\overline{q}(\xi,\tau)d\xi$ |
| $\overline{V}_s(\xi,\tau)$ | $-\int_1^{\xi}\overline{q}(\xi,\tau)d\xi+\frac{1}{1+\overline{m}_c}(\xi-1)\int_0^1\overline{q}(\xi,\tau)d\xi$ |
| $\overline{y}_i(\xi)$ | $C\left\{\frac{\sin\beta_i}{\beta_i}\cosh\alpha_i\xi-\frac{\sinh\alpha_i}{\alpha_i}\cos\beta_i\xi\right.$<br>$\left.-\overline{m}_c\frac{(\beta_i^2-k_i^2k_s^2)}{(\alpha_i^2+\beta_i^2)}(\cosh\alpha_i+\gamma_i\cos\beta_i)\left[\cos\beta_i\xi-\cosh\alpha_i\xi+\Lambda_i\left(\sin\beta_i\xi-\frac{\alpha_i}{\beta_i}\frac{1}{\gamma_i}\sinh\alpha_i\xi\right)\right]\right\}$ |

TABLE II.—RESPONSE OF A UNIFORM TIMOSHENKO BEAM TO A GENERAL LOAD—Continued

(a) Symmetrical free-free beam with a concentrated mass—Concluded

| Quantity | Analytical expression |
|---|---|
| $\psi_i(\xi)$ | $C\left(\frac{\beta_i^2-k_i^2k_s^2}{\beta_i}\right)\Big\{\gamma_i\frac{\beta_i}{\alpha_i}\frac{\sin\beta_i}{\beta_i}\sinh\alpha_i\xi+\frac{\sinh\alpha_i}{\alpha_i}\sin\beta_i\xi$ $-\overline{m}_c\frac{(\beta_i^2-k_i^2k_s^2)}{(\alpha_i^2+\beta_i^2)}(\cosh\alpha_i+\gamma_i\cos\beta_i)\left[\Lambda_i(\cos\beta_i\xi-\cosh\alpha_i\xi)-\left(\sin\beta_i\xi+\frac{\beta_i}{\alpha_i}\gamma_i\sinh\alpha_i\xi\right)\right]\Big\}$ |
| $\overline{M}_i(\xi)$ | $-C(\beta_i^2-k_i^2k_s^2)\Big\{\gamma_i\frac{\sin\beta_i}{\beta_i}\cosh\alpha_i\xi+\frac{\sinh\alpha_i}{\alpha_i}\cos\beta_i\xi$ $+\overline{m}_c\frac{(\beta_i^2-k_i^2k_s^2)}{(\alpha_i^2+\beta_i^2)}(\cosh\alpha_i+\gamma_i\cos\beta_i)\left[\Lambda_i\left(\sin\beta_i\xi+\frac{\alpha_i}{\beta_i}\sinh\alpha_i\xi\right)+(\cos\beta_i\xi+\gamma_i\cosh\alpha_i\xi)\right]\Big\}$ |
| $\overline{V}_i(\xi)$ | $C\frac{k_i^2}{\beta_i}\Big\{-\frac{\beta_i}{\alpha_i}\frac{\sin\beta_i}{\beta_i}\sinh\alpha_i\xi+\frac{\sinh\alpha_i}{\alpha_i}\sin\beta_i\xi$ $-\overline{m}_c\frac{(\beta_i^2-k_i^2k_s^2)}{(\alpha_i^2+\beta_i^2)}(\cosh\alpha_i+\gamma_i\cos\beta_i)\left[\Lambda_i\left(\cos\beta_i\xi+\frac{1}{\gamma_i}\cosh\alpha_i\xi\right)-\left(\sin\beta_i\xi-\frac{\beta_i}{\alpha_i}\sinh\alpha_i\xi\right)\right]\Big\}$ |
| $\Lambda_i$ | $\dfrac{\sin\beta_i-\frac{\beta_i}{\alpha_i}\sinh\alpha_i}{\cos\beta_i+\frac{1}{\gamma_i}\cosh\alpha_i}$ |
| $\gamma_i$ | $\dfrac{\alpha_i^2+k_i^2k_s^2}{\beta_i^2-k_i^2k_s^2}$ |
| $\alpha_i$ | $k_i\sqrt{\frac{1}{2}\left[-(k_s^2+k_{RI}^2)+\sqrt{(k_s^2-k_{RI}^2)^2+\frac{4}{k_i^2}}\right]}$ |
| $\beta_i$ | $k_i\sqrt{\frac{1}{2}\left[(k_s^2+k_{RI}^2)+\sqrt{(k_s^2-k_{RI}^2)^2+\frac{4}{k_i^2}}\right]}$ |

Frequency equation:

$$\frac{\alpha}{\beta}\gamma\sin\beta\cosh\alpha+\cos\beta\sinh\alpha+\overline{m}_c\frac{\alpha(\alpha^2+k^2k_s^2)}{\alpha^2+\beta^2}\left[2+\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)\sin\beta\sinh\alpha+\left(\gamma+\frac{1}{\gamma}\right)\cos\beta\cosh\alpha\right]=0$$

TABLE II.—RESPONSE OF A UNIFORM TIMOSHENKO BEAM TO A GENERAL LOAD—Continued

(b) Cantilever beam

| Quantity | Analytical expression |
|---|---|
| $\overline{y}(\xi,\tau)$ | $\overline{y}_s(\xi,\tau)+\sum_{i=1}^{\infty}\phi_i(\tau)\overline{y}_i(\xi)$ |
| $\psi(\xi,\tau)$ | $\psi_s(\xi,\tau)+\sum_{i=1}^{\infty}\phi_i(\tau)\psi_i(\xi)$ |
| $\overline{M}(\xi,\tau)$ | $\overline{M}_s(\xi,\tau)+\sum_{i=1}^{\infty}\phi_i(\tau)\overline{M}_i(\xi)$ |
| $\overline{V}(\xi,\tau)$ | $\overline{V}_s(\xi,\tau)+\sum_{i=1}^{\infty}\phi_i(\tau)\overline{V}_i(\xi)$ |
| $\phi_i(\tau)$ | $-\frac{P_i(\tau)}{m_i}+\frac{k_i}{m_i}\int_0^\tau P_i(\theta)\sin k_i(\tau-\theta)d\theta$ |
| $P_i(\tau)$ | $\frac{1}{k_1^2}\int_0^1\overline{q}(\xi,\tau)\overline{y}_i(\xi)d\xi$ |
| $m_i$ | $\int_0^1[\overline{y}_i^2(\xi)+k_{RI}^2\psi_i^2(\xi)]\,d\xi=-\frac{1}{2k_i}\psi_i(1)\left[\frac{\partial}{\partial k}\psi'(1)\right]_{k=k_i}$ |
| $\overline{y}_s(\xi,\tau)$ | $\int_0^\xi\int_0^\xi\int_1^\xi\int_1^\xi\overline{q}(\xi,\tau)(d\xi)^4-k_s^2\int_0^\xi\int_1^\xi\overline{q}(\xi,\tau)(d\xi)^2$ |
| $\psi_s(\xi,\tau)$ | $\int_0^\xi\int_1^\xi\int_1^\xi\overline{q}(\xi,\tau)(d\xi)^3$ |
| $\overline{M}_s(\xi,\tau)$ | $-\int_1^\xi\int_1^\xi\overline{q}(\xi,\tau)(d\xi)^2$ |
| $\overline{V}_s(\xi,\tau)$ | $-\int_1^\xi\overline{q}(\xi,\tau)d\xi$ |
| $\overline{y}_i(\xi)$ | $C\left[\cos\beta_i\xi-\cosh\alpha_i\xi+\Lambda_i\left(\sin\beta_i\xi-\frac{\alpha_i}{\beta_i}\frac{1}{\gamma_i}\sinh\alpha_i\xi\right)\right]$ |
| $\psi_i(\xi)$ | $C\frac{\beta_i^2-k_i^2k_s^2}{\beta_i}\left[\Lambda_i(\cos\beta_i\xi-\cosh\alpha_i\xi)-\left(\sin\beta_i\xi+\frac{\beta_i}{\alpha_i}\gamma_i\sinh\alpha_i\xi\right)\right]$ |
| $\overline{M}_i(\xi)$ | $C(\beta_i^2-k_i^2k_s^2)\left[\Lambda_i\left(\sin\beta_i\xi+\frac{\alpha_i}{\beta_i}\sinh\alpha_i\xi\right)+(\cos\beta_i\xi+\gamma_i\cosh\alpha_i\xi)\right]$ |
| $\overline{V}_i(\xi)$ | $C\frac{k_i^2}{\beta_i}\left[\Lambda_i\left(\cos\beta_i\xi+\frac{1}{\gamma_i}\cosh\alpha_i\xi\right)-\left(\sin\beta_i\xi-\frac{\beta_i}{\alpha_i}\sinh\alpha_i\xi\right)\right]$ |
| $\Lambda_i$ | $\frac{\sin\beta_i-\frac{\beta_i}{\alpha_i}\sinh\alpha_i}{\cos\beta_i+\frac{1}{\gamma_i}\cosh\alpha_i}$ |
| $\gamma_i$ | $\frac{\alpha_i^2+k_i^2k_s^2}{\beta_i^2-k_i^2k_s^2}$ |
| $\alpha_i$ | $k_i\sqrt{\frac{1}{2}\left[-(k_s^2+k_{RI}^2)+\sqrt{(k_s^2-k_{RI}^2)^2+\frac{4}{k_i^2}}\right]}$ |
| $\beta_i$ | $k_i\sqrt{\frac{1}{2}\left[(k_s^2+k_{RI}^2)+\sqrt{(k_s^2-k_{RI}^2)^2+\frac{4}{k_i^2}}\right]}$ |

Frequency equation:

$$2+\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)\sin\beta\sinh\alpha+\left(\gamma+\frac{1}{\gamma}\right)\cos\beta\cosh\alpha=0$$

TABLE II.—RESPONSE OF A UNIFORM TIMOSHENKO BEAM TO A GENERAL LOAD—Concluded

(c) Simply supported beam

| Quantity | Analytical expression |
|---|---|
| $\bar{y}(\xi,\tau)$ | $\bar{y}_s(\xi,\tau)+\sum_{i=1}^{\infty}\left\{[\phi_i(\tau)]_{k_i=a_i}+[\phi_i(\tau)]_{k_i=b_i}\right\}\sin i\pi\xi$ |
| $\psi(\xi,\tau)$ | $\psi_s(\xi,\tau)+\sum_{i=1}^{\infty}i\pi\left\{\left(1-\frac{k_s^2a_i^2}{i^2\pi^2}\right)[\phi_i(\tau)]_{k_i=a_i}+\left(1-\frac{k_s^2b_i^2}{i^2\pi^2}\right)[\phi_i(\tau)]_{k_i=b_i}\right\}\cos i\pi\xi$ |
| $\bar{M}(\xi,\tau)$ | $\bar{M}_s(\xi,\tau)+\sum_{i=1}^{\infty}i^2\pi^2\left\{\left(1-\frac{k_s^2a_i^2}{i^2\pi^2}\right)[\phi_i(\tau)]_{k_i=a_i}+\left(1-\frac{k_s^2b_i^2}{i^2\pi^2}\right)[\phi_i(\tau)]_{k_i=b_i}\right\}\sin i\pi\xi$ |
| $\bar{V}(\xi,\tau)$ | $\bar{V}_s(\xi,\tau)+\sum_{i=1}^{\infty}\frac{1}{i\pi}\left\{a_i^2[\phi_i(\tau)]_{k_i=a_i}+b_i^2[\phi_i(\tau)]_{k_i=b_i}\right\}\cos i\pi\xi$ |
| $\phi_i(\tau)$ | $-\frac{P_i(\tau)}{m_i}+\frac{k_i}{m_i}\int_0^{\tau}P_i(\theta)\sin k_i(\tau-\theta)d\theta$ |
| $P_i(\tau)$ | $\frac{1}{k_i^2}\int_0^1\bar{q}(\xi,\tau)\sin i\pi\xi\,d\xi$ |
| $m_i$ | $\frac{1}{2}\left[1+i^2\pi^2k_{RI}^2\left(1-\frac{k_s^2k_i^2}{i^2\pi^2}\right)^2\right]$ |
| $\bar{y}_s(\xi,\tau)$ | $\int_0^{\xi}\int_0^{\xi}\int_0^{\xi}\int_0^{\xi}\bar{q}(\xi,\tau)(d\xi)^4-k_s^2\int_0^{\xi}\int_0^{\xi}\bar{q}(\xi,\tau)(d\xi)^2-\xi\int_0^1\int_0^{\xi}\int_0^{\xi}\int_0^{\xi}\bar{q}(\xi,\tau)(d\xi)^4$ $-\left[\frac{\xi^3}{6}-\left(k_s^2+\frac{1}{6}\right)\xi\right]\int_0^1\int_0^{\xi}\bar{q}(\xi,\tau)(d\xi)^2$ |
| $\psi_s(\xi,\tau)$ | $\int_0^{\xi}\int_0^{\xi}\int_0^{\xi}\bar{q}(\xi,\tau)(d\xi)^3-\int_0^1\int_0^{\xi}\int_0^{\xi}\int_0^{\xi}\bar{q}(\xi,\tau)(d\xi)^4-\left(\frac{\xi^2}{2}-\frac{1}{6}\right)\int_0^1\int_0^{\xi}\bar{q}(\xi,\tau)(d\xi)^2$ |
| $\bar{M}_s(\xi,\tau)$ | $-\int_0^{\xi}\int_0^{\xi}\bar{q}(\xi,\tau)(d\xi)^2+\xi\int_0^1\int_0^{\xi}\bar{q}(\xi,\tau)(d\xi)^2$ |
| $\bar{V}_s(\xi,\tau)$ | $-\int_0^{\xi}\bar{q}(\xi,\tau)d\xi+\int_0^1\int_0^{\xi}\bar{q}(\xi,\tau)(d\xi)^2$ |

Frequency equations:

$k_i=a_i$ or $b_i$, where

$$a_i=\frac{1}{k_sk_{RI}}\sqrt{\frac{1}{2}\left\{1+i^2\pi^2(k_s^2+k_{RI}^2)-\sqrt{[1+i^2\pi^2(k_s^2+k_{RI}^2)]^2-4k_s^2k_{RI}^2i^4\pi^4}\right\}}$$

$$b_i=\frac{1}{k_sk_{RI}}\sqrt{\frac{1}{2}\left\{1+i^2\pi^2(k_s^2+k_{RI}^2)+\sqrt{[1+i^2\pi^2(k_s^2+k_{RI}^2)]^2-4k_s^2k_{RI}^2i^4\pi^4}\right\}}$$

The beam is divided arbitrarily into segments $\Delta x$ (see fig. 2), and the time interval is taken according to $\Delta t=\frac{1}{c_1}\Delta x$. This specifies a lattice of points in the space-time plane at the intersection of the I+ and I− characteristic lines. Consider a general point 1 (fig. 2) from which characteristics of both families have been drawn backwards in time. The II+ and II− lines have steeper slopes (since $c_2<c_1$) and terminate at points 2′ and 4′. Then the differentials in equations (50) may be replaced by the appropriate finite differences and the following equations are obtained:

$$\frac{1}{c_1}(M_1-M_2)+mr^2(\Omega_1-\Omega_2)-\frac{\Delta t}{2}(V_1+V_2)=0 \qquad (51a)$$

$$\frac{1}{c_1}(M_1-M_4)-mr^2(\Omega_1-\Omega_4)+\frac{\Delta t}{2}(V_1+V_4)=0 \qquad (51b)$$

$$\frac{1}{c_2}(V_1-V_{2'})-m(r_1-r_{2'})+mc_2\frac{\Delta t}{2}(\Omega_1+\Omega_{2'})+\frac{\Delta t}{2}(q_1+q_{2'})=0 \qquad (51c)$$

$$\frac{1}{c_2}(V_1-V_{4'})+m(r_1-r_{4'})+mc_2\frac{\Delta t}{2}(\Omega_1+\Omega_{4'})-\frac{\Delta t}{2}(q_1+q_{4'})=0 \qquad (51d)$$

It is assumed that $M$, $V$, $r$, and $\Omega$ are known at points 2, 3, and 4. Parabolic interpolation formulas are substituted into equations (51c) and (51d) to give the quantities at points 2′ and 4′ in terms of their values at points 2, 3, and 4. Then equations (51) become four equations for the four unknown quantities $M_1$, $V_1$, $r_1$, and $\Omega_1$ in terms of known values of $M$, $V$, $r$, and $\Omega$ at each of the lattice points 2, 3, and 4. These may be solved to obtain a matrix recurrence formula; however, a simplification may be introduced based on the fact that quantities at points 2 and 4 have already been determined to satisfy the following equations:

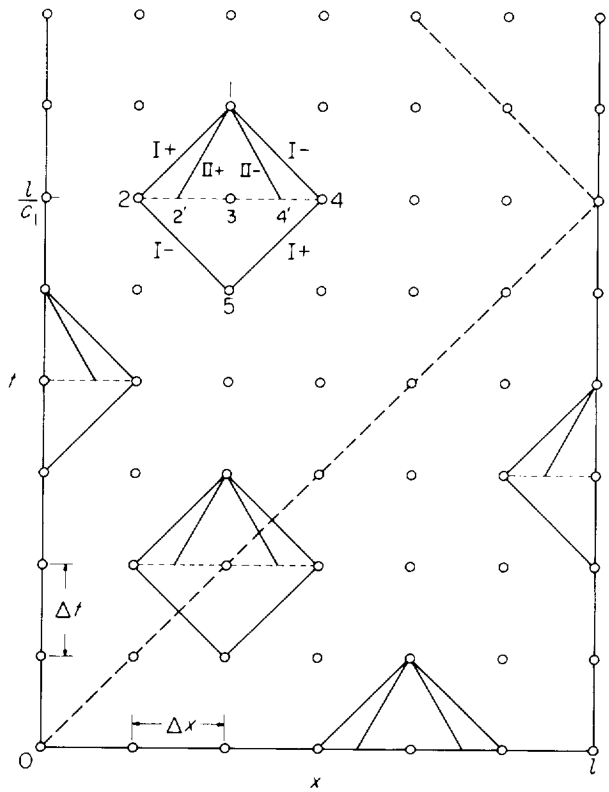



FIGURE 2. —Grid scheme for traveling-wave numerical procedure.

$$\left.\begin{aligned}\frac{1}{c_1}(M_2-M_5)-mr^2(\Omega_2-\Omega_5)+\frac{\Delta t}{2}(V_2+V_5)=0\\ \frac{1}{c_1}(M_4-M_5)+mr^2(\Omega_4-\Omega_5)-\frac{\Delta t}{2}(V_4+V_5)=0\end{aligned}\right\} \qquad (52)$$

Combining equations (52) with equations (51a) and (51b) leads to the result

$$mr^2(\Omega_1-\Omega_2-\Omega_4+\Omega_5)-\frac{\Delta t}{2}(V_1-V_5)=0 \qquad (53)$$

where $M$ has been eliminated. Now equations (51c), (51d), and (53) constitute three equations for the three unknowns $\Omega_1$, $V_1$, and $r_1$ in terms of known values of $\Omega$, $V$, and $r$ at each of the points 2, 3, 4, and 5. Solution of these equations yields the recurrence formula

$$\begin{vmatrix}\overline{\Omega}_1\\ \overline{V}_1\\ \overline{r}_1\end{vmatrix}=\frac{1}{1+K^2}\left\{[A_2]\begin{vmatrix}\overline{\Omega}_2\\ \overline{V}_2\\ \overline{r}_2\end{vmatrix}+[A_3]\begin{vmatrix}\overline{\Omega}_3\\ \overline{V}_3\\ \overline{r}_3\end{vmatrix}+[A_4]\begin{vmatrix}\overline{\Omega}_4\\ \overline{V}_4\\ \overline{r}_4\end{vmatrix}+[A_5]\begin{vmatrix}\overline{\Omega}_5\\ \overline{V}_5\\ \overline{r}_5\end{vmatrix}+[Q]\begin{vmatrix}\overline{q}_1\\ \overline{q}_2\\ \overline{q}_3\\ \overline{q}_4\end{vmatrix}\right\} \qquad (54)$$

where the quantities have been made dimensionless $\left(\overline{\Omega}\equiv\frac{\partial\psi}{\partial\tau}=\Omega l^2\sqrt{\frac{m}{EI}}\text{ and }\overline{v}\equiv\frac{\partial\overline{y}}{\partial\tau}=vl\sqrt{\frac{m}{EI}}\right)$ and where

$$[A_2]=\frac{1}{2}\begin{bmatrix}\left(2-K^2\frac{c_2^2}{c_1^2}\right) & \frac{c_2}{c_1}K & -\frac{c_2}{c_1}\frac{K}{k_{RI}} \\ -\left(2+\frac{c_2^2}{c_1^2}\right)\frac{c_2}{c_1}K & \frac{c_2^2}{c_1^2} & -\frac{c_2^2}{c_1^2}\frac{1}{k_{RI}} \\ \frac{c_2}{c_1}k_{RI}K(1+K^2) & -k_{RI}(1+K^2) & \frac{c_2^2}{c_1^2}(1+K^2)\end{bmatrix}$$

$$[A_3]=\left(1-\frac{c_2^2}{c_1^2}\right)\begin{bmatrix}-K^2 & \frac{c_1}{c_2}K & 0 \\ -\frac{c_2}{c_1}K & 1 & 0 \\ 0 & 0 & 1+K^2\end{bmatrix}$$

$$[A_4]=\frac{1}{2}\begin{bmatrix}\left(2-K^2\frac{c_2^2}{c_1^2}\right) & \frac{c_2}{c_1}K & \frac{c_2}{c_1}\frac{K}{k_{RI}} \\ -\left(2+\frac{c_2^2}{c_1^2}\right)\frac{c_2}{c_1}K & \frac{c_2^2}{c_1^2} & \frac{c_2^2}{c_1^2}\frac{1}{k_{RI}} \\ -\frac{c_2}{c_1}k_{RI}K(1+K^2) & k_{RI}(1+K^2) & \frac{c_2^2}{c_1^2}(1+K^2)\end{bmatrix}$$

$$[A_5]=\begin{bmatrix}-1 & -\frac{c_1}{c_2}K & 0 \\ \frac{c_2}{c_1}K & K^2 & 0 \\ 0 & 0 & 0\end{bmatrix}$$

$$[Q]=\begin{bmatrix}0 & -\frac{1}{2}k_{RI}K^2 & 0 & \frac{1}{2}k_{RI}K^2 \\ 0 & -\frac{1}{2}k_{RI}\frac{c_2}{c_1}K & 0 & \frac{1}{2}k_{RI}\frac{c_2}{c_1}K \\ k_{RI}^2\frac{c_1}{c_2}K(1+K^2) & \frac{1}{2}k_{RI}^2\frac{c_2}{c_1}K(1+K^2) & \left(1-\frac{c_2^2}{c_1^2}\right)\frac{c_1}{c_2}k_{RI}^2K(1+K^2) & \frac{1}{2}k_{RI}^2\frac{c_2}{c_1}K(1+K^2)\end{bmatrix}$$

$$K=\frac{1}{k_{RI}^2}\frac{c_2}{c_1}\frac{\Delta t}{2}$$

Note that the more appropriate parameter $\frac{c_1}{c_2}\left(=\frac{k_s}{k_{RI}}\right)$ has been used here instead of $k_s$.

The response of a beam may now be obtained by the repeated application of equation (54) except that, as is indicated in figure 2, special formulas which take into account the particular boundary and initial

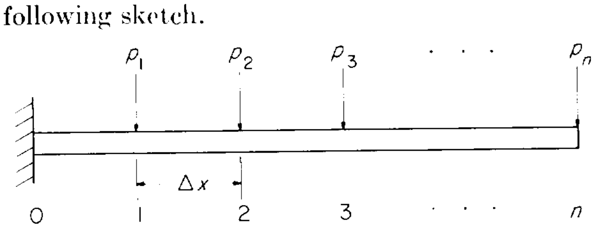
conditions of the problem, must be derived for boundary and initial points. In addition, it must be remembered that the characteristic lines are possible loci of discontinuities in the dependent variables or their derivatives. (See, for example, ref. 6.) Such discontinuities will arise, for example, if there are concentrated loads (or imposed velocities) that have histories which are discontinuous or have discontinuous derivatives. Discontinuities in $M$ or $\Omega$ are propagated with velocity $c_1$ (the locus of such a discontinuity is shown by the dashed characteristic lines in fig. 2); discontinuities in $V$ and $v$ propagate with velocity $c_2$. The magnitudes of discontinuities may be predetermined through the application of equations (50) in a manner which is illustrated in reference 6. Thus, in general, discontinuities may be added as they are encountered in the step-by-step solution. In the scheme which led to equation (54), it will be noted (fig. 2) that special consideration is necessary for points just above the locus of a discontinuity. Special formulas are certainly required where there is a jump in one of the functions and it may be desirable to account also for discontinuous first derivatives.

Once $\overline{\Omega}$, $\overline{V}$, and $\overline{v}$ have been determined at a point, $\overline{M}$ may be obtained at that point by integrating equation (50a) or equation (50b) along the proper characteristic from some boundary where $M$ is known.

It should be mentioned that the selection of the I+ and I− lines as the basic network is based on (rather intuitive) considerations of the stability of the numerical procedure with regard to propagation of errors. (See ref. 12.) It is assumed that the domain of dependence imposed by a numerical procedure should at least encompass the total theoretical domain of dependence. This would not be the case if the steeper II+ and II− characteristic lines were utilized as a basic grid.

### HOUBOLT'S METHOD

A cantilever beam acted upon by a series of concentrated loads $p_1, p_2, \ldots p_n$, is shown in the following sketch.




Such a beam has the deflection

$$|y_c| = [G]|p| \tag{55}$$

where the subscript $c$ is used here to indicate that the deflections are cantilever deflections measured with respect to station 0 and where $[G]$ is a matrix of stiffness influence coefficients. The inverse equation is

$$|p| = [G]^{-1}|y_c| \tag{56}$$

The deflections of a free-free beam may be expressed in terms of the cantilever influence coefficients $[G]$. For the free-free beam the symmetric deflection is $y = y_0 + y_c$ where $y_0$ is the deflection at station zero (the center of the free-free beam); hence, equation (56) becomes

$$|p| = [G]^{-1}|y| - y_0[G]^{-1}\left|\begin{matrix}1\\1\\ \cdot\\ \cdot\\ \cdot\\1\end{matrix}\right| \tag{57}$$

But, on the free-free beam, there is the additional load $p_0$. From the condition of overall equilibrium, $p_0$ is given by

$$p_0 = -\sum_{i=1}^{n} p_i = -\lfloor 1 \;\; 1 \;\; . \;\; . \;\; . \;\; 1 \rfloor [G]^{-1}|y| + y_0 \lfloor 1 \;\; 1 \;\; . \;\; . \;\; . \;\; 1 \rfloor [G]^{-1}\left|\begin{matrix}1\\1\\ \cdot\\ \cdot\\ \cdot\\1\end{matrix}\right| \tag{58}$$

and equations (57) and (58) may be combined into a single matrix equation for the loads at stations 0 to $n$ of a free-free beam. This equation may be written

$$|p| = [A]|y| \tag{59}$$

where

$$[A] = \begin{bmatrix} a & \lfloor b \rfloor \\ |b| & [G]^{-1} \end{bmatrix}$$

$$|b| = -[G]^{-1}\left|\begin{matrix}1\\1\\ \cdot\\ \cdot\\ \cdot\\1\end{matrix}\right|$$

$$\lfloor b \rfloor = -\lfloor 1 \;\; 1 \;\; . \;\; . \;\; . \;\; 1 \rfloor [G]^{-1}$$

$$a=[1\ 1\ .\ .\ .\ 1][G]^{-1}\begin{vmatrix}1\\1\\ \cdot\\ \cdot\\ \cdot\\1\end{vmatrix}$$

and where the vectors $|p|$ and $|y|$ now contain $p_0$ and $y_0$ terms, respectively.

Consider the symmetric motion of a free-free beam subjected to the applied distributed load $q(x,t)$. The equivalent concentrated loads at the stations 0, 1, . . . $n$ are, at any time,

$$|p|=[W](|q|-[m]|\ddot{y}|) \tag{60}$$

where $[m]$ is the diagonal mass matrix, and $[W]$ is the weighting matrix (ref. 13):

$$[W]=\frac{\Delta x}{24}\begin{bmatrix}7 & 6 & -1 & & & & \\ 2 & 20 & 2 & & & & \\ & 2 & 20 & 2 & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & 2 & 20 & 2 \\ & & & & & 2 & 20 & 2 \\ & & & & & -1 & 6 & 7\end{bmatrix}$$

The essential feature of the Houbolt procedure (ref. 8) is the method of expressing the second time derivative $\ddot{y}$. The acceleration at time $t_j=j\Delta t$ is written as follows:

$$|\ddot{y}|_j=\frac{1}{(\Delta t)^2}(2|y|_j-5|y|_{j-1}+4|y|_{j-2}-|y|_{j-3}) \tag{61}$$

and is obtained by passing a third-degree curve through the points at $t=t_j$, $t_{j-1}$, $t_{j-2}$, and $t_{j-3}$ at each station $x_i$. Substituting equations (60) and (61) into equation (59) (written for time $t=t_j$) and solving for $y_j$ leads to the recurrence equation

$$|y|_j=[B]|q|_j+\frac{1}{(\Delta t)^2}[C](5|y|_{j-1}-4|y|_{j-2}+|y|_{j-3}) \tag{62}$$

where

$$[B]=\left[[A]+\frac{2}{(\Delta t)^2}[W][m]\right]^{-1}[W]$$

$$[C]=[B][m]$$

Thus, the Houbolt method simultaneously determines all the deflections $y(x_i,t_j)$ (where $i=0$, 1, . . . $n$) in terms of the deflections $y(x_i,t_{j-1})$, $y(x_i,t_{j-2})$, and $y(x_i,t_{j-3})$. A concentrated load $2f(t)$ at the center of the free-free beam may be included in equation (62) by adding to the right-hand side the term

$$f_j|d| \tag{63}$$

where

$$|d|=\left[[A]+\frac{2}{(\Delta t)^2}[W][m]\right]^{-1}\begin{vmatrix}1\\0\\0\\ \cdot\\ \cdot\\ \cdot\\0\end{vmatrix}$$

The solution of a problem by repeated application of equation (62) requires that the initial conditions of the problem at time $t=0$ be expressed in terms of fictitious ordinates $|y|_{-1}, |y|_{-2}$ at times $t_{-1}$ and $t_{-2}$. This is accomplished by expressing the time derivatives $|\dot{y}|_0$ and $|\ddot{y}|_0$ in terms of a third-degree curve passed through points at $t=t_1$, 0, $t_{-1}$, and $t_{-2}$ at every station. For the case where the beam is initially at rest and the applied loads increase continuously from zero initial values, the initial conditions

$$|y|_0=|\dot{y}|_0=|\ddot{y}|_0=0 \tag{64}$$

are converted by this process to

$$\left.\begin{aligned}|y|_0&=0\\ |y|_{-1}&=-|y|_1\\ |y|_{-2}&=-8|y|_1\end{aligned}\right\} \tag{65}$$

Then, equation (62), applied for $j=1$, yields

$$|y|_1=[B]|q|_1-\frac{4}{(\Delta t)^2}[C]|y|_1$$

which may be solved for $\lfloor y \rfloor_1$ to obtain

$$\lfloor y \rfloor_1 = \left[ [I] + \frac{4}{(\Delta t)^2} [C] \right]^{-1} [B] \lfloor q \rfloor_1 \tag{66}$$

The application of equation (62) for $j=2, 3, \ldots$ is now straightforward. All that remains is the determination of the cantilever influence coefficients $[G]$.

In this connection it should be pointed out that, although no restrictions have been made on the beam theory to be used, the application of the method, as formulated, with Timoshenko's theory requires that the deflections $y$ be interpreted in a general sense and include also the rotations of the cross sections $\psi$. Thus, with two quantities to be determined at each station, the order of the matrices is $2n$ and the required computational labor is roughly four times that required with the use of the elementary beam theory. A compromise which affords increased accuracy over the elementary theory, yet avoids this large increase in computational labor, is the use of a theory which contains transverse shear freedom but no rotary inertia. With no associated inertia loading, the rotations $\psi$ need not be explicitly included in the step-by-step dynamic analysis and do not appear in the recurrence formula, equation (62). The cantilever influence coefficients are determined as follows on the basis of this latter theory.

The influence function (Green's function) $G(x;x_1)$ is the solution $y$ of the equations

$$\left.\begin{aligned} &\frac{\partial}{\partial x} EI \frac{\partial \psi}{\partial x} + A_s G\left(\frac{\partial y}{\partial x} - \psi\right) = 0 \\ &\frac{\partial}{\partial x}\left[A_s G\left(\frac{\partial y}{\partial x} - \psi\right)\right] = -\delta(x-x_1) \end{aligned}\right\} \tag{67}$$

and the boundary conditions

$$y(0) = \psi(0) = \frac{\partial \psi}{\partial x}(l) = \frac{\partial y}{\partial x}(l) - \psi(l) = 0 \tag{68}$$

However, the deflection may be written as the sum of bending and shear contributions, $y = y_B + y_S$ with $\psi = \frac{\partial y_B}{\partial x}$, and it is expedient to write for the influence function

$$G(x;x_1) = G_B(x;x_1) + G_S(x;x_1) \tag{69}$$

where $G_B$ and $G_S$ are the solutions of the differential equations (equivalent to eqs. (67))

$$\left.\begin{aligned} &\frac{\partial^2}{\partial x^2} EI \frac{\partial^2}{\partial x^2} G_B = \delta(x-x_1) \\ &\frac{\partial}{\partial x} A_s G \frac{\partial}{\partial x} G_S = -\delta(x-x_1) \end{aligned}\right\} \tag{70}$$

and boundary conditions (equivalent to eqs. (68))

$$\left.\begin{aligned} G_B(0;x_1) = \frac{\partial G_B}{\partial x}(0;x_1) = \frac{\partial^2 G_B}{\partial x^2}(l;x_1) \\ = \frac{\partial}{\partial x} EI \frac{\partial^2 G_B}{\partial x^2}(l;x_1) = 0 \\ G_S(0;x_1) = \frac{\partial G_S}{\partial x}(l;x_1) = 0 \end{aligned}\right\} \tag{71}$$

For given distributions of bending and shear stiffnesses, equations (70) may be integrated directly in conjunction with the boundary conditions (eqs. (71)). For a uniform beam, the resulting total influence functions are:

$$\left.\begin{aligned} G(x;x_1) &= \frac{1}{A_s G} x + \frac{1}{EI}\left(\frac{x^2 x_1}{2} - \frac{x^3}{6}\right) \qquad (x < x_1) \\ &= \frac{1}{A_s G} x_1 + \frac{1}{EI}\left(\frac{x_1^2 x}{2} - \frac{x_1^3}{6}\right) \qquad (x > x_1) \end{aligned}\right\} \tag{72}$$

In dimensionless terms, equations (72) may be written

$$\left.\begin{aligned} \overline{G}(\xi;\xi_1) &= k_s^2 \xi + \frac{\xi^2}{2}\left(\xi_1 - \frac{\xi}{3}\right) \qquad (\xi < \xi_1) \\ &= k_s^2 \xi_1 + \frac{\xi_1^2}{2}\left(\xi - \frac{\xi_1}{3}\right) \qquad (\xi > \xi_1) \end{aligned}\right\} \tag{73}$$

and, for a uniform beam, the recurrence formula, equation (62), becomes

$$\lfloor \overline{y} \rfloor_j = [\overline{B}] \lfloor \overline{q} \rfloor_j + \frac{1}{(\Delta \tau)^2} [\overline{B}][I](5\lfloor \overline{y} \rfloor_{j-1} - 4\lfloor \overline{y} \rfloor_{j-2} + \lfloor \overline{y} \rfloor_{j-3}) \tag{74}$$

where

$$[\overline{B}] = \left[ [\overline{A}] + \frac{2}{(\Delta \tau)^2} [\overline{W}][I] \right]^{-1} [\overline{W}]$$

$$[\overline{A}] = \begin{bmatrix} \overline{a} & \lfloor \overline{b} \rfloor \\ \lfloor \overline{b} \rfloor & [\overline{G}]^{-1} \end{bmatrix}$$

$$\overline{a}=\lfloor 1\ 1\ .\ .\ .\ 1\rfloor[\overline{G}]^{-1}\begin{Bmatrix}1\\1\\ \cdot\\ \cdot\\ \cdot\\1\end{Bmatrix}$$

$$\{\overline{b}\}=-[\overline{G}]^{-1}\begin{Bmatrix}1\\1\\ \cdot\\ \cdot\\ \cdot\\1\end{Bmatrix}$$

$$\lfloor\overline{b}\rfloor=-\lfloor 1\ 1\ .\ .\ .\ 1\rfloor[\overline{G}]^{-1}$$

$$[\overline{W}]=\frac{\Delta\xi}{24}\begin{bmatrix}7 & 6 & -1 & & & & \\ 2 & 20 & 2 & & & & \\ & 2 & 20 & 2 & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & 2 & 20 & 2 \\ & & & & & 2 & 20 & 2 \\ & & & & & -1 & 6 & 7\end{bmatrix}$$

The elements $\overline{G}_{ij}$ of the matrix of influence coefficients $[\overline{G}]$ are seen to be

$$\overline{G}_{ij}=k_s^2 i\Delta\xi+\frac{i^2}{2}\left(j-\frac{i}{3}\right)(\Delta\xi)^3 \qquad (i<j)$$

$$=k_s^2 j\Delta\xi+\frac{j^2}{2}\left(i-\frac{j}{3}\right)(\Delta\xi)^3 \qquad (i>j)$$

where $i$ designates the row and $j$, the column.

## RESULTS AND DISCUSSION

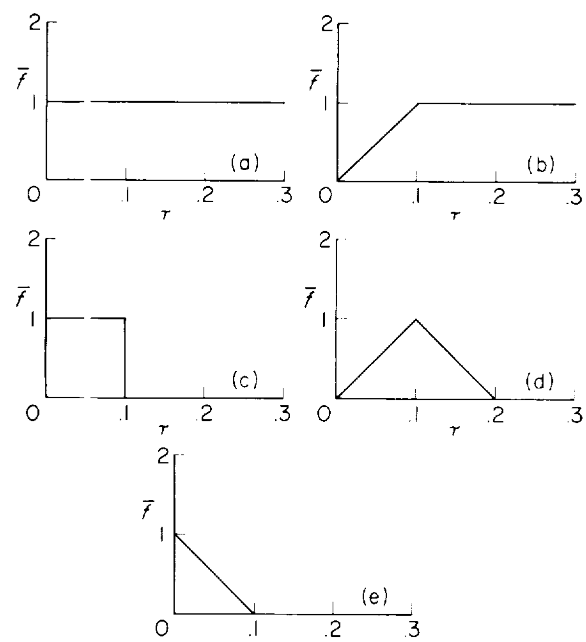
### MODAL SOLUTIONS

For illustrative purposes, example computations have been made for the case of a uniform free-free beam, for which $k_{RI}=0.1$, $k_s=0.2$, and $\overline{m}_c=0$. The beam is subjected to the applied concentrated load

$$\overline{q}(\xi,\tau)=2\overline{f}(\tau)\delta(\xi)$$

where $\overline{f}(\tau)$ has each of the time variations shown in figure 3. The given values of $k_{RI}$ and $k_s$ are appropriate to a beam having a solid rectangular cross section and a ratio of half-length $l$ to depth equal to 2.887. The calculated response has been limited to the history of transverse shear at the point $\xi=\frac{1}{2}$. Moment calculations are omitted because they do not provide as severe a test of the analytical methods. The point $\xi=\frac{1}{2}$ is chosen arbitrarily, since the location corresponding to maximum transverse shear is not known in advance.

The responses to the step and ramp-platform loads were obtained on the basis of elementary theory from table I(a) and on the basis of Timoshenko's theory from table II(a). In each case, six modes were used in the expansions. The responses to the other three functions (figs. 3(c), 3(d), and 3(e)) were obtained by superposition of the step and ramp-platform results. The resulting shear histories are shown in figures 4, 5, 6, 7, and 8 up to a time corresponding approximately to the period of the first natural mode of vibration of the beam.




(a) Step function. (b) Ramp-platform function.
(c) Square pulse. (d) Triangular pulse.
(e) Blast pulse.

FIGURE 3.—Some fundamental load histories.

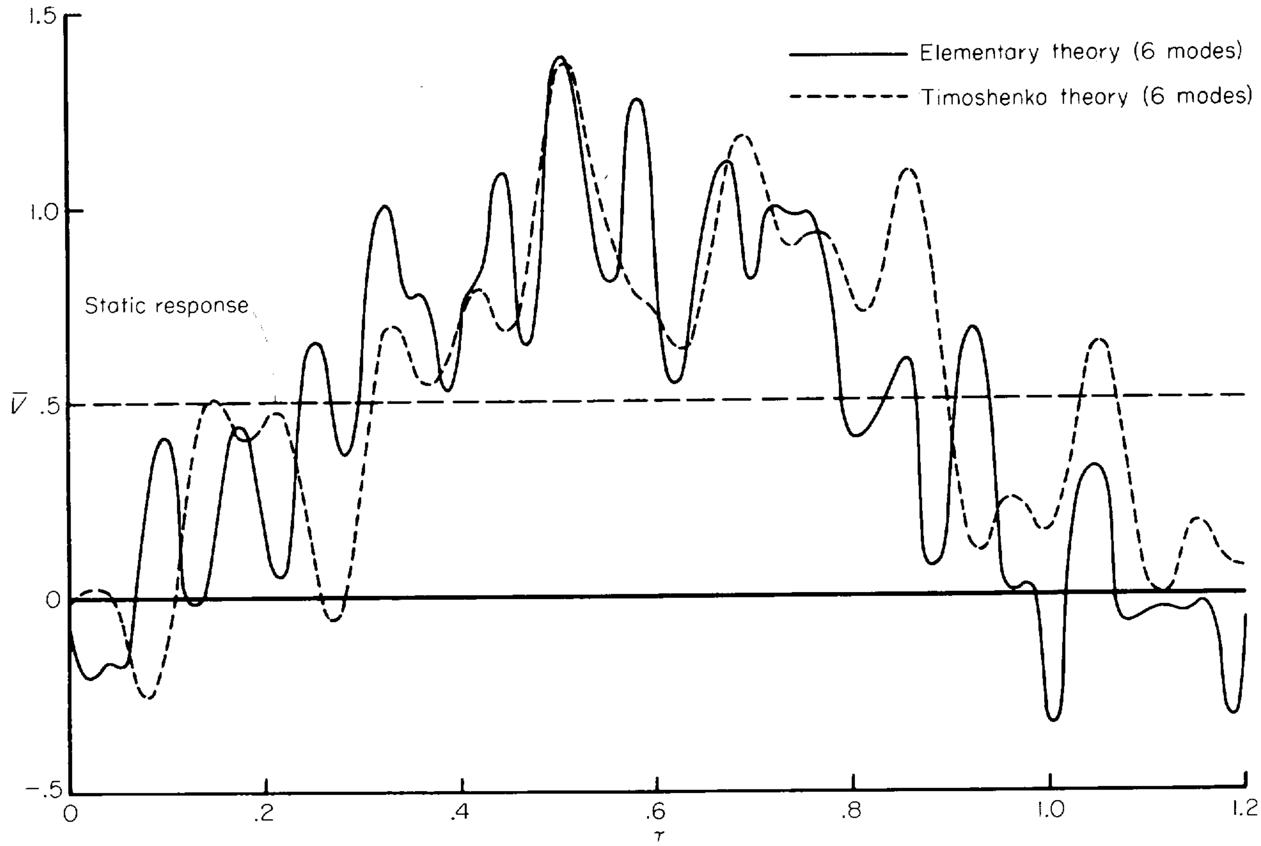



FIGURE 4.—Shear response $\bar{V}\left(\frac{1}{2},\tau\right)$ of a uniform free-free beam to a step load concentrated at $\xi=0$.

The long dashed curve in each figure is the static portion of the response. Thus, the largest dynamic overshoot factor, achieved by the step loading (fig. 4), is approximately 2.7.

It will be noted that the blast pulse load (fig. 3(e)) has only one-half as much impulse as that contained in the square and triangular pulse loads (figs. 3(c) and 3(d)). Hence, if the responses to the three pulse loads (see figs. 6, 7, and 8) are to be compared on the basis of equal input impulse, the response to the blast pulse must be doubled. On this basis, the blast pulse is seen to cause the highest shear stress at $\xi=\frac{1}{2}$.

An indication of the convergence of the modal results in figures 4 to 8 is provided by the bar graphs in figure 9. On each graph, the heights of the bars correspond to the magnitude of the static portion of the response (zero frequency) and to the amplitudes of the terms in the series expansion for the dynamic portion. (For each load, $\tau$ is sufficiently large so that the load function has attained its constant value.) The bars, thus, represent the maximum possible contribution of each term to the total. Note that, for the loads of long duration (the step and ramp-platform functions), the static part and the first term contribute a proportionately large share of the response and sufficient accuracy could be obtained with only three modes. On the other hand, the adverse effect of reducing the load duration is illustrated by the bar graphs for the responses to to the three pulse loads. In each case, no convergence is apparent for the first few modes and, in the cases of the square and blast pulses, there is some doubt as to the adequacy of the even six modes, particularly with the use of the elementary theory.

Further evidence of the effect of load duration on convergence and, in addition, an indication of the effect of load distribution are given by the following cases of the response of a uniform simply supported beam to various loads:

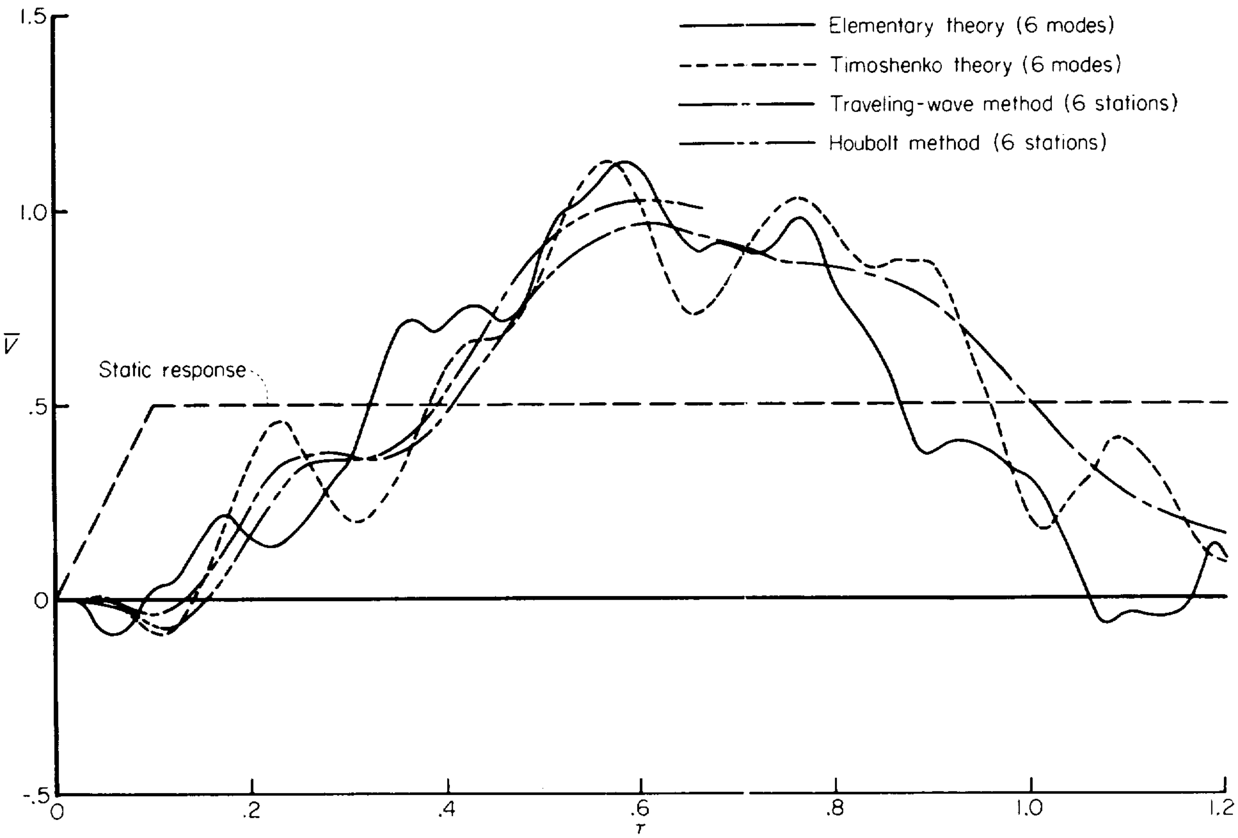



FIGURE 5.—Shear response $\bar{V}\left(\frac{1}{2},\tau\right)$ of a uniform free-free beam to a ramp-platform load concentrated at $\xi=0$.

Case (1): For a uniformly distributed step load $\bar{q}(\xi,\tau)=\boldsymbol{1}(\tau)$,

$$\bar{M}(\xi,\tau)=-\frac{\xi^2}{2}+\frac{\xi}{2}-\frac{4}{\pi^3}\sum_{i=1,3,\ldots}^{\infty}\frac{1}{i^3}\sin i\pi\xi\cos i^2\pi^2\tau^2$$

$$\bar{V}(\xi,\tau)=-\xi+\frac{1}{2}-\frac{4}{\pi^2}\sum_{i=1,3,\ldots}^{\infty}\frac{1}{i^2}\cos i\pi\xi\cos i^2\pi^2\tau^2$$

Case (2): For a step load concentrated at the center $\bar{q}(\xi,\tau)=\delta\left(\xi-\frac{1}{2}\right)\boldsymbol{1}(\tau)$,

$$\bar{M}(\xi,\tau)=\frac{\xi}{2}-\left(\xi-\frac{1}{2}\right)\boldsymbol{1}\left(\xi-\frac{1}{2}\right)-\frac{2}{\pi^2}\sum_{i=1,3,\ldots}^{\infty}(-1)^{\frac{i-1}{2}}\frac{1}{i^2}\sin i\pi\xi\cos i^2\pi^2\tau$$

$$\bar{V}(\xi,\tau)=\frac{1}{2}-\boldsymbol{1}\left(\xi-\frac{1}{2}\right)-\frac{2}{\pi}\sum_{i=1,3,\ldots}^{\infty}(-1)^{\frac{i-1}{2}}\frac{1}{i}\cos i\pi\xi\cos i^2\pi^2\tau$$

Case (3): For a uniformly distributed impulse load $\bar{q}(\xi\ \tau)=\delta(\tau)$,

$$\bar{M}(\xi,\tau)=\frac{4}{\pi}\sum_{i=1,3,\ldots}^{\infty}\frac{1}{i}\sin i\pi\xi\sin i^2\pi^2\tau$$

$$\bar{V}(\xi,\tau)=4\sum_{i=1,3,\ldots}^{\infty}\cos i\pi\xi\sin i^2\pi^2\tau$$

Case (4): For an impulse load concentrated at the center $\bar{q}(\xi,\tau)=\delta\left(\xi-\frac{1}{2}\right)\delta(\tau)$,

$$\bar{M}\ \xi,\tau)=2\sum_{i=1,3,\ldots}^{\infty}(-1)^{\frac{i-1}{2}}\sin i\pi\xi\sin i^2\pi^2\tau$$

$$\bar{V}(\xi,\tau)=2\pi\sum_{i=1,3,\ldots}^{\infty}(-1)^{\frac{i-1}{2}}i\cos i\pi\xi\sin i^2\pi^2\tau$$

These results are based on elementary beam theory. (The last two cases are simply time derivatives of the first two cases.) The effect of load duration is illustrated by comparing cases (1) and (3) (or

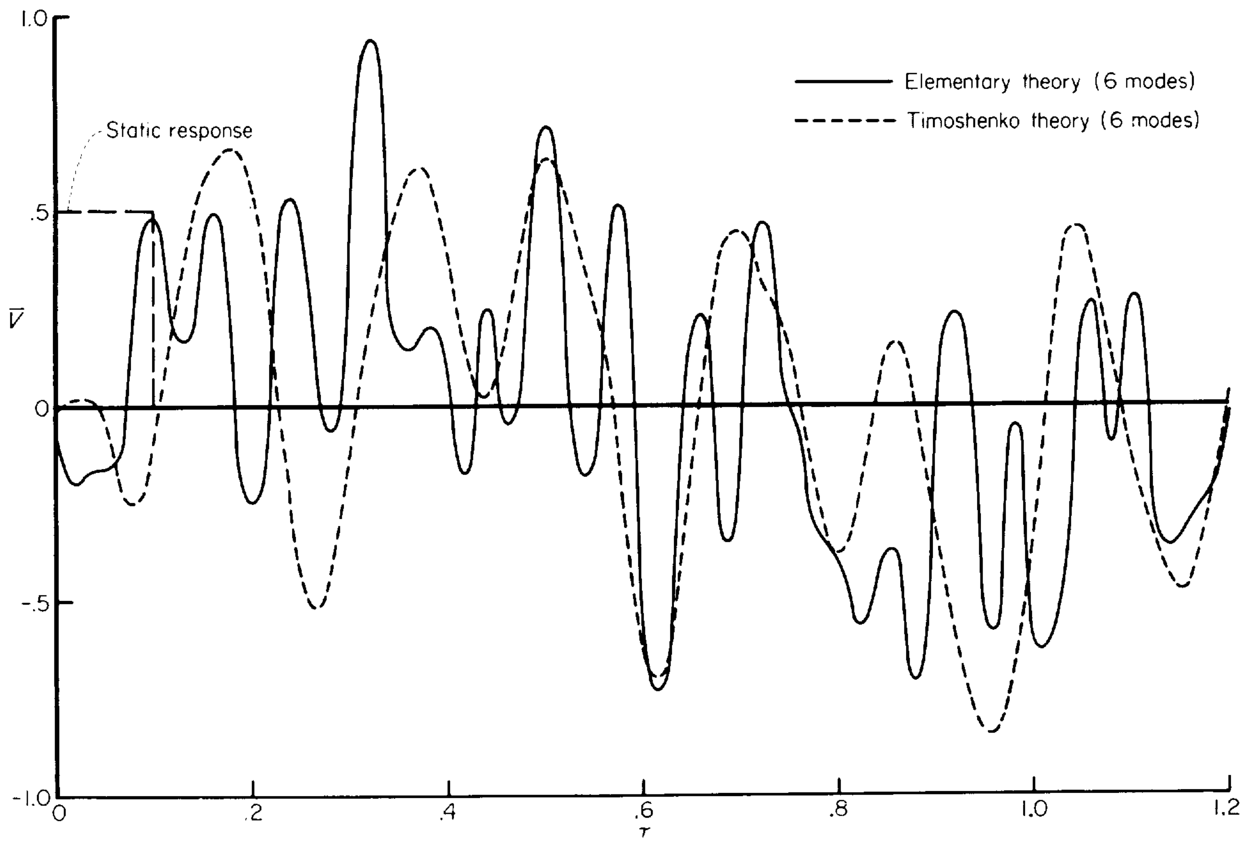



FIGURE 6.—Shear response $\bar{V}\left(\frac{1}{2},\tau\right)$ of a uniform free-free beam to a square pulse load concentrated at $\xi=0$.

cases (2) and (4)). Changing from a load of infinite duration (the step load) to a load of zero duration (the impulse) introduces a factor $i^2$ and, hence, reduces the rate of convergence (and, in fact, produces divergent series in both case (3) and case (4)). Similarly, the effect of spatial distribution may be seen by comparing cases (1) and (2) (or cases (3) and (4)). Changing from a distributed to a concentrated load introduces a factor $i$ and hence reduces the rate of convergence. The apparent change in the sign of half the terms is not significant since each series is essentially an irregularly alternating series (except at certain specific combinations of $\xi$ and $\tau$).

### THE NEED FOR TIMOSHENKO'S THEORY

The question of which theory should be used to determine the response of a beam to a transient load is intimately related to the convergence of the result. This is because the secondary effects of transverse shear and rotary inertia become increasingly important for the higher modes. Note in figure 9, for example, the growing disparity between the natural frequencies of a uniform free-free beam calculated on the basis of the elementary and Timoshenko theories. Thus, if it is determined that, for a given beam subjected to a certain load, modes strongly affected by transverse shear and rotary inertia contribute a large share of the response, it is unlikely that elementary theory will yield correct results. In case the given beam is a complicated nonuniform structure, a rational procedure for determining the proper theory would be to consider a uniform approximation to the given beam, quickly obtain the response of the uniform beam to the given load by elementary theory (table I) and investigate the convergence of the response, and at the same time to consider the influence of rotary inertia and transverse shear on the modes (as manifested by the differences in natural frequencies obtained with the elementary and Timoshenko theories).

This reasoning is generally, though not conclusively, confirmed by the results in figures 4 to 9.

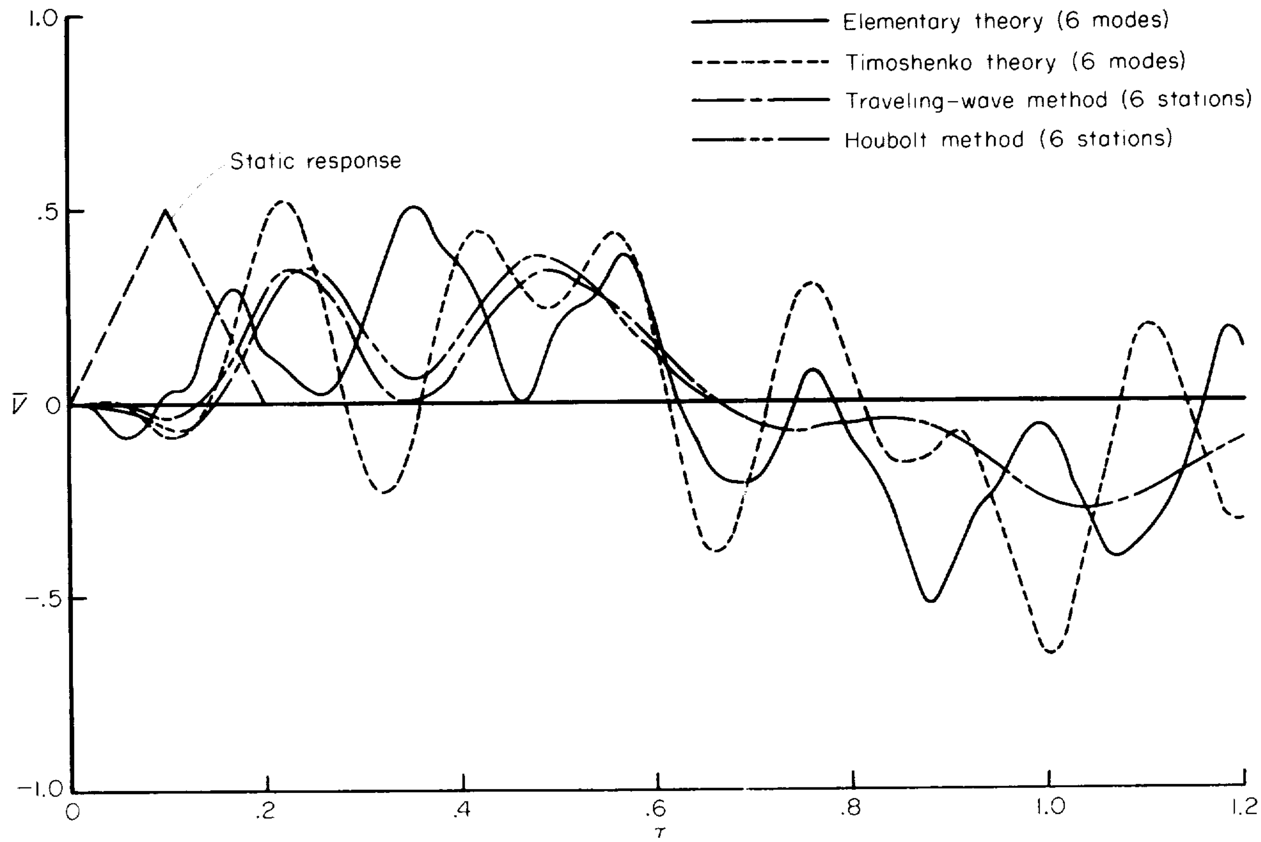



FIGURE 7. Shear response $\bar{V}\left(\frac{1}{2},\tau\right)$ of a uniform free-free beam to a triangular pulse load concentrated at $\xi=0$.

Only the first mode of the free-free beam shows good agreement between the frequencies as given by the elementary and Timoshenko theories. (See fig. 9.) Thus, the responses to the step and ramp-platform loads (figs. 4 and 5), which depend heavily on the first mode and the static contribution, also show good agreement between the two theories. The responses to the pulse loads (figs. 6, 7, and 8), obtained with the two different theories, bear little or no resemblance to each other since they depend heavily on the higher modes. However, except for the response to the square pulse load (fig. 6), the two theories do yield about the same peak stress. (It is felt that the positive peak achieved in the first half period should be given more weight than the negative peak achieved later, since the latter would be considerably diminished by material damping which has not been included in this analysis.)

Boundary conditions also influence the need for a more refined theory. For example, although the elementary theory is adequate for obtaining the shear due to a step load on the free-free beam of figure 4, it cannot be used to obtain the shear at the center of the same beam where the input is a prescribed "step-velocity" of the point $\xi=0$ (the so-called problem of the "instantaneous arrest of the root of a moving cantilever beam"). In the latter case, elementary theory yields a divergent series (ref. 14), whereas Timoshenko's theory yields a modal solution which converges to finite values (ref. 6).

## NUMERICAL SOLUTIONS

The two numerical procedures, the traveling-wave method and Houbolt's method, have also been used to calculate the transient response of the uniform free-free beam considered in the previous sections to an applied ramp-platform load. In the calculations by both procedures, the beam was divided into six segments ($\Delta\xi=0.1667$). Results are shown in figure 5.

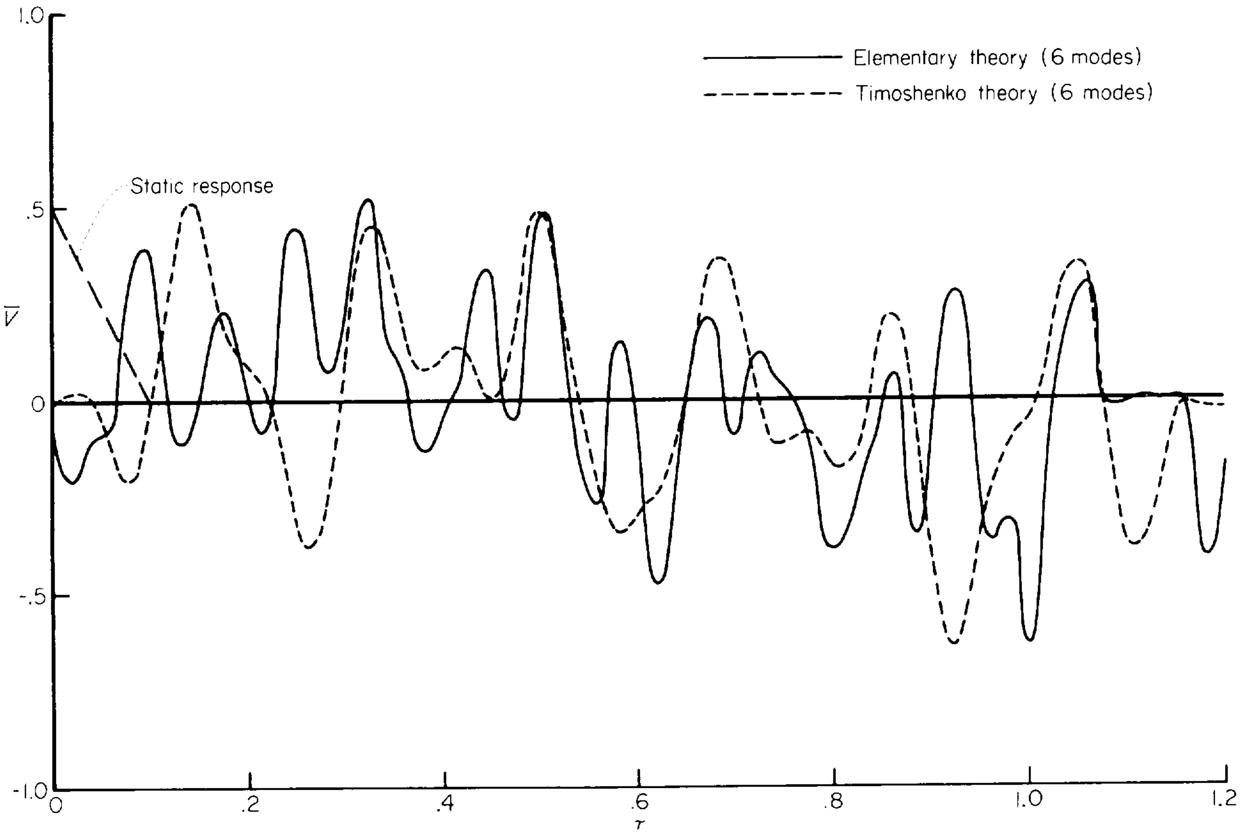



FIGURE 8.—Shear response $\bar{V}\left(\frac{1}{2},\tau\right)$ of a uniform free-free beam to a blast pulse load concentrated at $\xi=0$.

For the traveling-wave method, the time interval is

$$\Delta\tau = k_{RI}\Delta\xi = 0.01667$$

Relatively simple boundary formulas (based on linear interpolation) were used in this case and discontinuities in the derivatives of $V$ and $r$, arising from the discontinuities in the slope of the ramp-platform function, were ignored.

In contrast with the traveling-wave method, the Houbolt method imposes no inherent restriction on the selection of the time interval $\Delta\tau$ in relation to the space interval $\Delta\xi$. The time interval may be taken as large as is consistent with the desired accuracy. This freedom has been utilized in that the calculations by the Houbolt method have been made with the time interval $\Delta\tau = 0.03333$, which is twice the time interval used with the traveling-wave method. In addition, the computations by the Houbolt method do not include the effects of rotary inertia and were stopped at a point just beyond the peak load.

It will be noted that the numerical results in figure 5 approximate the (essentially converged) Timoshenko modal solution fairly well. Both numerical methods underestimate the peak stress, the traveling-wave method by 14 percent and the Houbolt method by 9 percent.

The greater accuracy of the Houbolt result is particularly significant since it was obtained with less computational labor due to the use of the larger time interval. It must be kept in mind, however, that economical use of the Houbolt method requires that it be applied in connection with the elementary theory or with the addition of transverse shear alone. If both rotary inertia and transverse shear must be included, the labor required in applying the Houbolt method is quadrupled. Fortunately, rotary inertia is negligible in many problems. (See, for example, ref. 15.)

It should be mentioned that the traveling-wave procedure, as so far conceived, has proved somewhat sensitive to minor changes in the scheme used to obtain recurrence formulas. For example,

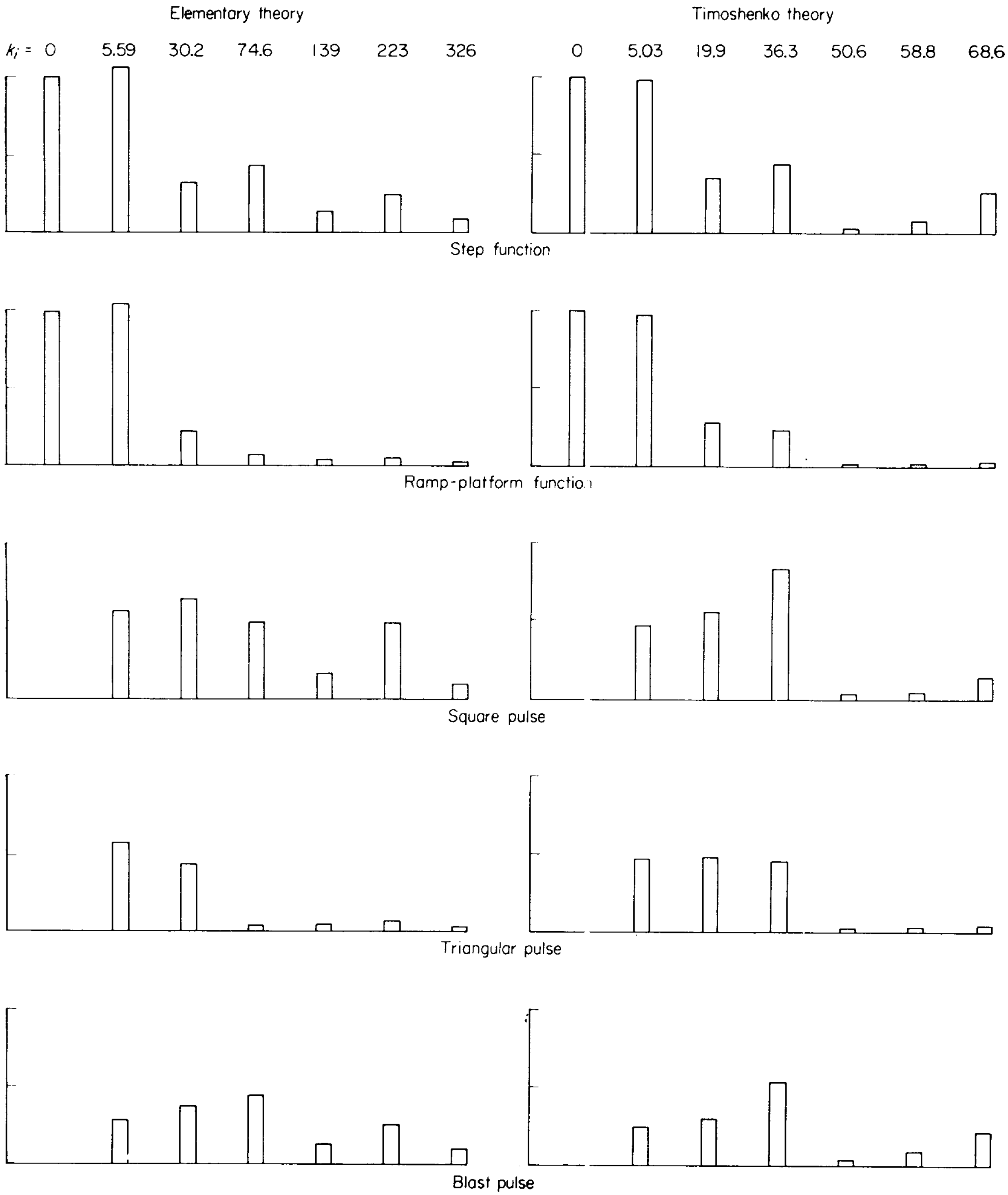



FIGURE 9.—Relative amplitudes of terms in expressions for $\overline{V}\left(\frac{1}{2},\tau\right)$.

the simplest and most obvious scheme (in which the II+ and II− characteristics are extended backward from the point 1 (fig. 2) to intermediate points on the I− and I+ characteristics connecting points 5 and 2 and points 5 and 4 and linear interpolation is used to determine the unknowns at these intermediate points) has been found to yield significantly less accurate results. A possible reason for this sensitivity stems from the fact that the slope of the basic network of characteristic lines is dependent on rotary inertia. In fact, rotary inertia is necessary to give the beam equations the wave character essential in the conception of a traveling-wave method. Thus, in view of the relative negligibility of rotary inertia for many practical problems, this sensitivity is perhaps not surprising. In general, it must be concluded that a traveling-wave numerical method of analysis which is superior to the Houbolt method has not yet been devised.

Numerical results have also been obtained for the response to the triangular pulse load by superposition of the ramp-platform results. These results are shown in figure 7. The results indicate that more degrees of freedom must be taken with both procedures to predict adequately the response of the beam to the given triangular pulse load, and from this example it appears that the modal method of solution is to be preferred. However, the simple problems discussed herein do not portray the main advantages of numerical methods. For example, numerical methods are readily extended to apply to nonuniform beams and conveniently adapted to the use of modern high-speed computers. A fundamental characteristic of numerical methods is the replacement of professional engineering time by routine computing time. Hence, numerical procedures are not to be condemned on the basis of the results in figure 7. The selection of the best method requires the consideration of all these factors in relation to the specific problem.

## CONCLUDING REMARKS

Williams type modal solutions, based on both the elementary and Timoshenko beam theories, have been given for the response of several uniform beams to a general transient load. The response to any specific load may be obtained from these solutions by performing a series of indicated direct integrations of the load function. Typical computed results have been shown for the shear response of a free-free beam to various concentrated loads.

The convergence of modal solutions is shown to depend both on the history and distribution of the load. Decreasing either the duration of the loading or the region over which the load is applied reduces the rate of convergence and may produce divergence.

The need for a more refined theory, as compared to elementary theory, is intimately related to the rate of convergence of the modal solution. If modes which are strongly dependent on transverse shear and rotary inertia contribute a large portion of the response, Timoshenko's theory must be used.

Comparison of the Houbolt and traveling-wave numerical methods indicates that the Houbolt procedure has many advantages over the traveling-wave procedure as so far conceived.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Field, Va., *February 5, 1958*.

# APPENDIX

## SYMMETRICAL NATURAL VIBRATION OF A UNIFORM FREE-FREE TIMOSHENKO BEAM WITH A CONCENTRATED MASS

### NATURAL MODES AND FREQUENCIES

The differential equations and boundary conditions governing symmetrical natural vibration of a free-free beam with a concentrated mass at its center may be written in the following dimensionless forms:

$$\left.\begin{aligned} &\psi'' + \frac{1}{k_s^2}(\bar{y}' - \psi) + k^2 k_{RI}^2 \psi = 0 \\ &\frac{1}{k_s^2}(\bar{y}' - \psi)' + k^2 \bar{y} = 0 \end{aligned}\right\} \tag{A1}$$

$$\psi(0) = 0 \tag{A2a}$$

$$\psi'(1) = 0 \tag{A2b}$$

$$\bar{y}'(1) - \psi(1) = 0 \tag{A2c}$$

$$\frac{1}{k_s^2}\,\bar{y}'(0) + \bar{m}_c k^2 \bar{y}(0) = 0 \tag{A2d}$$

where $\bar{y}(\xi); \psi(\xi)$ is the natural mode and $k$ is proportional to the circular frequency of vibration. Each of the solutions of equations (A1) has the form

$$\bar{y}(\xi) = A e^{\lambda\xi}$$

$$\psi(\xi) = B e^{\lambda\xi}$$

where $A$ and $B$ are arbitrary constants. Substituting this form into equations (A1) leads to the biquadratic equation

$$\lambda^4 + k^2(k_s^2 + k_{RI}^2)\lambda^2 + k^2(k^2 k_s^2 k_{RI}^2 - 1) = 0 \tag{A3}$$

and to the following relationship between $A$ and $B$:

$$B = \frac{\lambda^2 + k^2 k_s^2}{\lambda} A \tag{A4}$$

Equation (A3) has the four solutions $\lambda = \pm\alpha$ and $\lambda = \pm i\beta$ where

$$\left.\begin{aligned} \alpha &= k\sqrt{\frac{1}{2}\left[-(k_s^2 + k_{RI}^2) + \sqrt{(k_s^2 - k_{RI}^2)^2 + \frac{4}{k^2}}\right]} \\ \beta &= k\sqrt{\frac{1}{2}\left[(k_s^2 + k_{RI}^2) + \sqrt{(k_s^2 - k_{RI}^2)^2 + \frac{4}{k^2}}\right]} \end{aligned}\right\} \tag{A5}$$

The general solution of equations (A1) may be written in the form

$$\left.\begin{aligned} \bar{y}(\xi) &= C_1 \cosh\alpha\xi + C_2 \sinh\alpha\xi + C_3 \cos\beta\xi + C_4 \sin\beta\xi \\ \psi(\xi) &= \frac{\alpha^2 + k^2 k_s^2}{\alpha}(C_2 \cosh\alpha\xi + C_1 \sinh\alpha\xi) \\ &\quad + \frac{\beta^2 - k^2 k_s^2}{\beta}(C_4 \cos\beta\xi - C_3 \sin\beta\xi) \end{aligned}\right\} \tag{A6}$$

where $C_1$, $C_2$, $C_3$, and $C_4$ are arbitrary functions of $k$ which must be determined in order to satisfy the boundary conditions (eqs. (A2)).

Substituting equations (A6) into equations (A2) yields four homogeneous algebraic equations for $C_1$, $C_2$, $C_3$, and $C_4$. The existence of nontrivial solutions of these equations requires the vanishing of the determinant of coefficients. This criterion yields the frequency equation

$$\begin{aligned} F(k) \equiv \frac{\alpha}{\beta}\,\gamma \sin\beta \cosh\alpha + \cos\beta \sinh\alpha \\ + \bar{m}_c \frac{\alpha(\alpha^2 + k^2 k_s^2)}{\alpha^2 + \beta^2}\left[2 + \left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)\sin\beta \sinh\alpha \right. \\ \left. + \left(\gamma + \frac{1}{\gamma}\right)\cos\beta \cosh\alpha\right] = 0 \end{aligned} \tag{A7}$$

where

$$\gamma = \frac{\alpha^2 + k^2 k_s^2}{\beta^2 - k^2 k_s^2}$$

The solutions of equation (A7) are the natural frequencies of vibration $k_i$, where $i = 0, 1, 2, \ldots$.

For $k=k_i$, the homogeneous algebraic equations may be solved for the relative magnitudes of the quantities $C_1$, $C_2$, $C_3$, and $C_4$. For $i=1, 2, \ldots$, the resulting vibration mode shapes may be written in the form

$$\left.\begin{aligned}
\overline{y}_i(\xi)=C\Bigg\{&\frac{\sin\beta_i}{\beta_i}\cosh\alpha_i\xi-\frac{\sinh\alpha_i}{\alpha_i}\cos\beta_i\xi-\overline{m}_c\frac{\beta_i^2-k_i^2k_s^2}{\alpha_i^2+\beta_i^2}(\cosh\alpha_i+\gamma_i\cos\beta_i)\Big[\cos\beta_i\xi-\cosh\alpha_i\xi\\
&+\Lambda_i\left(\sin\beta_i\xi-\frac{\alpha_i}{\beta_i}\frac{1}{\gamma_i}\sinh\alpha_i\xi\right)\Big]\Bigg\}\qquad(i=1,2,\ldots)\\
\psi_i(\xi)=C\frac{\beta_i^2-k_i^2k_s^2}{\beta_i}\Bigg\{&\frac{\beta_i}{\alpha_i}\gamma_i\frac{\sin\beta_i}{\beta_i}\sinh\alpha_i\xi+\frac{\sinh\alpha_i}{\alpha_i}\sin\beta_i\xi-\overline{m}_c\frac{\beta_i^2-k_i^2k_s^2}{\alpha_i^2+\beta_i^2}(\cosh\alpha_i\\
&+\gamma_i\cos\beta_i)\Big[\Lambda_i(\cos\beta_i\xi-\cosh\alpha_i\xi)-\left(\sin\beta_i\xi+\frac{\beta_i}{\alpha_i}\gamma_i\sinh\alpha_i\xi\right)\Big]\Bigg\}\qquad(i=1,2,\ldots)
\end{aligned}\right\}\quad\text{(A8)}$$

where

$$\Lambda_i=\frac{\sin\beta_i-\dfrac{\beta_i}{\alpha_i}\sinh\alpha_i}{\cos\beta_i+\dfrac{1}{\gamma_i}\cosh\alpha_i}$$

The rigid body mode, corresponding to $k_0=0$, has the components $\overline{y}_0(\xi)=C$ and $\psi_0(\xi)=0$.

#### ORTHOGONALITY OF THE NATURAL MODES

The differential equations (A1) are satisfied by any of the infinite number of natural modes and corresponding frequencies. Thus, for the $i$th mode,

$$\psi_i''+\frac{1}{k_s^2}(\overline{y}_i'-\psi_i)+k_i^2k_{RI}^2\psi_i=0$$

$$\frac{1}{k_s^2}(\overline{y}_i'-\psi_i)'+k_i^2\overline{y}_i=0$$

Let the first of these equations be multiplied by the rotational component $\psi_j$ of the $j$th mode and the second by the translational component $\overline{y}_j$. If the resulting equations are added and integrated over the beam length, there results

$$k_i^2\int_0^1(\overline{y}_i\overline{y}_j+k_{RI}^2\psi_i\psi_j)d\xi=-\int_0^1\frac{1}{k_s^2}(\overline{y}_i'-\psi_i)'\overline{y}_jd\xi-\int_0^1\psi_i''\psi_jd\xi-\int_0^1\frac{1}{k_s^2}(\overline{y}_i'-\psi_i)\psi_jd\xi$$

Integrating by parts the first two integrals on the right-hand side of the preceding equation yields:

$$k_i^2\int_0^1(\overline{y}_i\overline{y}_j+k_{RI}^2\psi_i\psi_j)d\xi=-\left[\frac{1}{k_s^2}(\overline{y}_i'-\psi_i)\overline{y}_j+\psi_i'\psi_j\right]_0^1+\int_0^1\frac{1}{k_s^2}(\overline{y}_i'-\psi_i)(\overline{y}_j'-\psi_j)d\xi+\int_0^1\psi_i'\psi_j'd\xi\quad\text{(A9)}$$

This process is valid also if the roles of the $i$th and $j$th modes are reversed. Interchanging $i$ and $j$ in equation (A9) and subtracting the result from equation (A9) leads to

$$(k_i^2-k_j^2)\int_0^1(\overline{y}_i\overline{y}_j+k_{RI}^2\psi_i\psi_j)d\xi=-\left[\frac{1}{k_s^2}(\overline{y}_i'-\psi_i)\overline{y}_j-\frac{1}{k_s^2}(\overline{y}_j'-\psi_j)\overline{y}_i+\psi_i'\psi_j-\psi_j'\psi_i\right]_0^1\quad\text{(A10)}$$

If now the boundary conditions (eqs. (A2)) are imposed, equation (A10) is found to reduce to

$$(k_i^2-k_j^2)\int_0^1(\overline{y}_i\overline{y}_j+k_{RI}^2\psi_i\psi_j)d\xi=-(k_i^2-k_j^2)\overline{m}_c\overline{y}_i(0)\overline{y}_j(0)$$

or

$$\int_0^1\left\{[1+\overline{m}_c\delta(\xi)]\overline{y}_i(\xi)\overline{y}_j(\xi)+k_{RI}^2\psi_i(\xi)\psi_j(\xi)\right\}d\xi=0\qquad(j\neq i)\quad\text{(A11)}$$

since

$$\int_0^1\overline{m}_c\delta(\xi)\overline{y}_i(\xi)\overline{y}_j(\xi)d\xi=\overline{m}_c\overline{y}_i(0)\overline{y}_j(0)$$

Equation (A11) is the orthogonality condition satisfied by the natural vibration modes of a uniform free-free Timoshenko beam with a concentrated mass at the center.

#### DETERMINATION OF THE GENERALIZED MASS

The determination of the generalized mass

$$m_i=\int_0^1\left\{[1+\overline{m}_c\delta(\xi)]\overline{y}_i^2(\xi)+k_{RI}^2\psi_i^2(\xi)\right\}d\xi$$

by direct integration is a somewhat laborious process for $i\neq 0$. Fortunately, $m_i$ can be expressed in terms of certain boundary values of the mode shapes by the application of a limiting process to equation (A10) in which the functions $\overline{y}_i$ and $\psi_i$ are considered as continuous functions of $k$. Thus, if

$$k_j=k_i+dk$$

and

$$\overline{y}_j=\overline{y}_i+d\overline{y}=\overline{y}_i+\frac{\partial\overline{y}}{\partial k}dk$$

$$\psi_j=\psi_i+d\psi=\psi_i+\frac{\partial\psi}{\partial k}dk$$

equation (A10) becomes, in the limit as $dk$ approaches zero,

$$\int_0^1[\overline{y}_i^2+k_{RI}^2\psi_i^2]d\xi=\frac{1}{2k_i}\left[\frac{1}{k_s^2}(\overline{y}_i'-\psi_i)\left(\frac{\partial\overline{y}}{\partial k}\right)_{k=k_i}-\frac{1}{k_s^2}\left(\frac{\partial}{\partial k}\overline{y}'-\frac{\partial\psi}{\partial k}\right)_{k=k_i}\overline{y}_i+\psi_i'\left(\frac{\partial\psi}{\partial k}\right)_{k=k_i}-\psi_i\left(\frac{\partial}{\partial k}\psi'\right)_{k=k_i}\right]_0^1 \quad (i=1, 2, \ldots) \tag{A12}$$

This equation is applicable to uniform beams and may be extended, if desired, to nonuniform beams.

On substitution of the boundary conditions (eqs (A2)), equation (A12) reduces to

$$\int_0^1[\overline{y}_i^2+k_{RI}^2\psi_i^2]d\xi=-\overline{m}_c\overline{y}_i^2(0)-\frac{1}{2k_i}\psi_i(1)\left[\frac{\partial}{\partial k}\psi'(1)\right]_{k=k_i} \qquad (i=1, 2, \ldots)$$

Hence, the generalized mass is given by

$$m_i=-\frac{1}{2k_i}\psi_i(1)\left[\frac{\partial}{\partial k}\psi'(1)\right]_{k=k_i} \qquad (i=1, 2, \ldots) \tag{A13}$$

Note that only the second boundary condition (eq. (A2b)) is altered by differentiation with respect to $k$. This arises from the fact that only this boundary condition depends for its satisfaction on the frequency equation (A7); that is, the mode shapes (eqs. (A8)) satisfy the other boundary conditions for any value of $k$ but satisfy equation (A2b) only for $k=k_i$ since

$$\psi'(1)=\frac{C}{\alpha}(\beta^2-k^2k_s^2)F(k)$$

where $F(k)$ is defined by equation (A7). It is only by virtue of this dependence of one or more boundary conditions on the frequency equation that equations (A12) yield a value of $m_i$. Hence, it must be concluded that, for a beam for which none of the boundary conditions depend for their satisfaction on the frequency equation (as, for example a simply supported beam), equations (A12) are not applicable. For such a beam, $m_i$ is determined by direct integration.

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